

## Gauging the Poisson sigma model

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**ABSTRACT:** We show how to carry out the gauging of the Poisson sigma model in an AKSZ inspired formulation by coupling it to a generalization of the Weil model worked out in ref. [31]. We call the resulting gauged field theory, Poisson-Weil sigma model. We study the BV cohomology of the model and show its relation to Hamiltonian basic and equivariant Poisson cohomology. As an application, we carry out the gauge fixing of the pure Weil model and of the Poisson-Weil model. In the first case, we obtain the 2-dimensional version of Donaldson-Witten topological gauge theory, describing the moduli space of flat connections on a closed surface. In the second case, we recover the gauged  $A$  topological sigma model worked out by Baptista describing the moduli space of solutions of the so-called vortex equations.

**KEYWORDS:** BRST Quantization, Sigma Models, Topological Field Theories, Differential and Algebraic Geometry.

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## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. The Weil sigma model</b>	<b>4</b>
<b>3. The gauge fixing of the Weil model</b>	<b>8</b>
<b>4. The Poisson-Weil sigma model</b>	<b>11</b>
<b>5. The gauge fixing of the Poisson-Weil model</b>	<b>18</b>
<b>6. Outlook</b>	<b>23</b>
<b>A. The Weil model in components</b>	<b>24</b>
<b>B. The Poisson-Weil model in components</b>	<b>25</b>
<b>C. <math>G</math> and general covariance</b>	<b>27</b>

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## 1. Introduction

One efficient way of generating a sigma model on a non trivial manifold  $X$  is the gauging of a sigma model on a simpler manifolds  $M$  carrying the action of a Lie group  $G$  such that  $X \simeq M/G$ . The target space of the gauged model turns out to be precisely  $X$ . In many interesting cases, a symplectic structure on  $M$  and a moment map for the  $G$ -action can be defined and this construction is a particular case of a general procedure called Hamiltonian reduction [1].

The usefulness of gauging sigma models was first recognized by Witten in [2], where the gauged linear sigma-model with target  $X = \mathbb{C}^n$  and group  $G = U(1)$  was used to study non-gauged sigma-models into weighted projective spaces and Calabi-Yau hypersurfaces thereof. Later, in [3], applying the same procedure, Witten employed a gauged linear sigma-model with target  $X = \mathbb{C}^{kn}$  and group  $G = U(k)$  in a study of the quantum cohomology of Grassmannians.

The study of gauged sigma models, however, was initiated long before Witten's work. Developing on the results of Gates, Hull and Roček in [4], the gauging of (2,2) supersymmetric sigma models on biHermitian manifolds was studied originally by Hull, Papadopoulos and Spence in [5]. Their analysis was however limited to the subclass of almost product structure target spaces because of the lack of an off-shell (2,2) supersymmetric action in the general case at that time. After the realization that biHermitian geometry is naturally framed in generalized complex and Kaehler geometry by Hitchin and Gualtieri [6, 7],

(2,2) supersymmetric sigma models have been fruitfully formulated in this new powerful geometric language. In this way, the off-shell (2,2) supersymmetric sigma model action on general biHermitian manifolds was recently obtained in ref. [8]. This has led the authors of ref. [9] to extend the analysis of [5] to general biHermitian target spaces. In [10], the same analysis has been carried out in the on-shell formalism.

(2,2) supersymmetric sigma models are rather complicated quantum field theories and, so, they are difficult to study. In 1988, Witten showed that a (2,2) supersymmetric sigma model on a Calabi-Yau manifold (a particular case of biHermitian manifold) could be twisted in two different ways, to give the so called  $A$  and  $B$  topological sigma models [11, 12]. Unlike the original untwisted sigma model, the topological models are soluble: the calculation of observables can be reduced to classical problems of geometry. Topological sigma models on general biHermitian manifolds have been worked out in recent years to a various degree of depth in [13–17]. However, only a small number of papers has been devoted to the study of gauged topological sigma models [18–20] and these are concerned with the Calabi-Yau case only. The problem arises of constructing gauged topological sigma models with more general biHermitian target space geometries.

In the last few years, many attempts have been made to construct topological sigma models with generalized complex and Kaehler target manifolds [21–26]. In [23–25], the sigma models were worked out by employing the Batalin-Vilkovisky (BV) quantization algorithm [27, 28] in the Alexandrov-Kontsevich-Schwartz-Zaboronsky (AKSZ) formulation [29]. To date, this seems to be the most promising approach to the solution of the problem, though, as shown in [30], the implementation of gauge fixing remains a major technical obstacle even in the simplest cases.

In ref. [31], we showed how Hamiltonian symmetry reduction could be incorporated in the sigma model on generalized complex manifolds worked out in refs. [23, 24] (the so-called Hitchin model). This was achieved by coupling the sigma model to a kind of ghostly Poisson sigma model called Weil model. To illustrate our procedure, we applied it also to the standard Poisson sigma model [32, 33] in the AKSZ formulation of refs. [34, 35].

As it turns out, coupling to the Weil model amounts to a gauging procedure. In [31], the background principal bundle was taken to be trivial. In this paper we show that this restriction is not in any way essential. With appropriate modifications, the Weil sigma model can be formulated and the coupling of the Weil model to the relevant sigma model can be implemented for a general principal bundle. We restrict ourselves to the Poisson sigma model for its simplicity and its independent interest. Our construction results in a gauged Poisson sigma model, which we call Poisson-Weil sigma model.

It is instructive to write down the classical action of the Poisson-Weil sigma model to see its relation to the conventional formulation of standard Poisson sigma model. The target space is a Poisson manifold  $M$  with Poisson structure  $P^{ab}$  carrying a Hamiltonian action of a Lie group  $G$  with fundamental vector field  $u_i$  and moment map  $\mu_i$  and leaving  $P^{ab}$  invariant. The base space  $\Sigma$  supports a principal  $G$ -bundle  $Q$ . The fields are an embedding field  $x^a$ , a cotangent space valued 1-form field  $\eta_a$ , as in the ordinary Poisson model, and a gauge field  $A^i$ , a coadjoint scalar field  $b_i$  and an adjoint scalar field  $B^{+i}$ . The

classical action is

$$S = \int_{\Sigma} \left[ -b_i B^{+i} - b_i F_A^i - \mu_i(x) B^{+i} + \eta_a D_A x^a + \frac{1}{2} P^{ab}(x) \eta_a \eta_b \right], \quad (1.1)$$

where

$$F_A^i = dA^i + \frac{1}{2} f^i{}_{jk} A^j A^k, \quad (1.2)$$

$$D_A x^a = dx^a - u_i^a(x) A^i \quad (1.3)$$

are the gauge curvature of  $A^i$  and the gauge covariant derivative of  $x^a$ , respectively. The Poisson-Weil sigma model enjoys a large symmetry which extends that of the ordinary Poisson sigma model by the gauge symmetry. The symmetry closes only on shell, as in the ordinary case. This disease is cured by using a suitable BV formulation generalizing that of [34, 35].

The Weil and Poisson-Weil sigma models have a very rich algebraic and geometric structure. The BV cohomology of the Weil model is related to the basic cohomology of the Weil algebra  $W(\mathfrak{g})$  of the Lie algebra  $\mathfrak{g}$  of  $G$  (in turn isomorphic to the de Rham cohomology of the classifying space  $BG$  of the group  $G$ ). The BV cohomology of the Poisson-Weil model is related to the Hamiltonian basic and equivariant Poisson cohomology of the Poisson manifold. To some extent, this is expected on general grounds and the fact that this is indeed so shows the soundness of the models.

As an application, we work out the gauge fixing of the pure Weil sigma model and of the Poisson-Weil sigma model in the BV framework. In the first case, we obtain the 2-dimensional version of Donaldson-Witten theory, a topological field theory describing the moduli space of flat connections on a closed surface [36, 37]. In the second case, we recover the gauged topological sigma model worked out by Baptista in refs. [18–20], which describes the moduli space of solutions of the so-called vortex equations and is a gauged version of Witten’s  $A$ -model [11, 12].

The plan of the paper is as follows. In section 2, we present a generalization of the Weil sigma model originally worked out in ref. [31], which is valid for a general principal  $G$ -bundle on the sigma model world sheet and is suitable for the constructions of the following sections. In section 3, we work out a gauge fixing of the Weil model and show that it yields the 2-dimensional version of Donaldson-Witten theory. In section 4, we formulate a generalization of the Poisson-Weil sigma model worked out in ref. [31] and show that it constitutes a gauging of the ordinary Poisson model. In section 5, we carry out the gauge fixing of the Poisson-Weil sigma model and show that it reproduces the gauged topological sigma model by Baptista. In section 6, we outline briefly potential generalizations of the constructions of this paper to the case where  $G$  is a Poisson-Lie group. Finally, in the appendices, we conveniently collect various relations and identities which may help the reader willing to check the details of our analysis.

### Acknowledgments

We thank F. Bonechi, T. Strobl, H. Bursztyn for helpful discussions. We thank also, J.-P. Ortega, R. L. Fernandes, V. Ginzburg and E. Meinrenken for correspondence.

## 2. The Weil sigma model

In this section, we present a generalization of the Weil sigma model originally worked out in ref. [31], which is suitable for our construction. Though the covariance of the superfields of the version of Weil sigma model studied here is more general, its BV formulation is essentially the same as that of [31]. The reader is therefore invited to read that paper for more details on the BV formalism used and the derivation of the classical master equation and BV variations below.

We consider a geometrical setting consisting of the following elements.

1. A closed surface  $\Sigma$ .
2. A compact connected Lie group  $G$  with Lie algebra  $\mathfrak{g}$ .
3. A principal  $G$ -bundle  $Q$  over  $\Sigma$ .

With  $\Sigma$  there is associated the degree shifted tangent bundle  $T[1]\Sigma$ . Let  $a_1 : T[1]\Sigma \rightarrow \Sigma$  be the associated bundle projection. Then, we can construct the pull-back principal bundle  $a_1^*Q$  over  $T[1]\Sigma$ . Concretely,  $a_1^*Q$  can be described in the language of 1-cocycles as follows. Let  $\{U_A\}$  be an open covering of  $\Sigma$  such that  $Q|_{U_A} \simeq U_A \times G$ . Let  $\{g_{AB}\}$  be the  $G$ -valued 1-cocycle representing  $Q$  with respect to the covering  $\{U_A\}$ . Define  $\mathbf{g}_{AB} = g_{AB} \circ a_1$ . Then,  $\{\mathbf{g}_{AB}\}$  is the  $G$ -valued 1-cocycle representing  $a_1^*Q$  with respect to the covering  $\{a_1^{-1}(U_A)\}$  of  $T[1]\Sigma$ .

A generalized connection  $\mathbf{c}$  of  $a_1^*Q$  is defined as follows.  $\mathbf{c}$  is given locally on each open set  $a_1^{-1}(U_A)$  of  $T[1]\Sigma$  as a function  $\mathbf{c}_A \in \Gamma(a_1^{-1}(U_A), \mathfrak{g}[1])$  with  $\mathbf{c}_A = \text{Ad } \mathbf{g}_{AB} \mathbf{c}_B - \mathbf{g}_{AB} \mathbf{d}(\mathbf{g}_{AB}^{-1})$  on  $a_1^{-1}(U_A) \cap a_1^{-1}(U_B) \neq \emptyset$ , where  $\mathbf{d}$  is the homological vector field of  $T[1]\Sigma$  corresponding to the de Rham differential  $d$  of  $\Sigma$ . (The choice of the sign of the affine term is conventional.) We denote by  $\text{Conn}(T[1]\Sigma, a_1^*Q)$  the affine space of generalized connections of  $a_1^*Q$ .

The adjoint and coadjoint bundles  $\text{Ad } a_1^*Q$ ,  $\text{Ad}^\vee a_1^*Q$  are defined as  $\text{Ad } a_1^*Q = a_1^*Q \times_G \mathfrak{g}$  and  $\text{Ad}^\vee a_1^*Q = a_1^*Q \times_G \mathfrak{g}^\vee$ . A section  $\mathbf{s} \in \Gamma(T[1]\Sigma, \text{Ad } a_1^*Q)$  is given locally on each open set  $a_1^{-1}(U_A)$  of  $T[1]\Sigma$  as a function  $\mathbf{s}_A \in \Gamma(a_1^{-1}(U_A), \mathfrak{g})$  with  $\mathbf{s}_A = \text{Ad } \mathbf{g}_{AB} \mathbf{s}_B$  on  $a_1^{-1}(U_A) \cap a_1^{-1}(U_B) \neq \emptyset$  and similarly for  $\text{Ad}^\vee a_1^*Q$ . Degree shifting is achieved by replacing  $\mathfrak{g}$  by  $\mathfrak{g}[n]$  above and similarly for  $\mathfrak{g}^\vee$ .

The field content of the Weil sigma model is the following.

1. A section  $\mathbf{b} \in \Gamma(T[1]\Sigma, \text{Ad}^\vee a_1^*Q[0])$ .
2. A section  $\mathbf{B} \in \Gamma(T[1]\Sigma, \text{Ad}^\vee a_1^*Q[-1])$ .
3. A generalized connection  $\mathbf{c} \in \text{Conn}(T[1]\Sigma, a_1^*Q)$ .
4. A section  $\mathbf{C} \in \Gamma(T[1]\Sigma, \text{Ad } a_1^*Q[2])$ .<sup>1</sup>

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<sup>1</sup>In [31]  $\mathbf{B}$   $\mathbf{C}$  were denoted by  $B$ ,  $\Gamma$ , respectively.

The BV odd symplectic form is given by

$$\Omega_W = \int_{T[1]\Sigma} \varrho \left[ \delta \mathbf{b}_i \delta \mathbf{c}^i + \delta \mathbf{B}_i \delta \mathbf{C}^i \right]. \quad (2.1)$$

The action of the Weil sigma model is given by

$$S_W = \int_{T[1]\Sigma} \varrho \left[ \mathbf{b}_i \left( d\mathbf{c}^i - \frac{1}{2} f^i_{jk} \mathbf{c}^j \mathbf{c}^k + \mathbf{C}^i \right) - \mathbf{B}_i (d\mathbf{C}^i - f^i_{jk} \mathbf{c}^j \mathbf{C}^k) \right]. \quad (2.2)$$

$S_W$  satisfies the classical BV master equation

$$(S_W, S_W)_W = 0, \quad (2.3)$$

where  $(\cdot, \cdot)_W$  are the BV antibrackets associated with the BV form  $\Omega_W$  [31].

The BV variations of the Weil sigma model fields are

$$\delta_W \mathbf{b}_i = d\mathbf{b}_i + f^k_{ji} \mathbf{c}^j \mathbf{b}_k + f^k_{ji} \mathbf{C}^j \mathbf{B}_k, \quad (2.4a)$$

$$\delta_W \mathbf{c}^i = d\mathbf{c}^i - \frac{1}{2} f^i_{jk} \mathbf{c}^j \mathbf{c}^k + \mathbf{C}^i, \quad (2.4b)$$

$$\delta_W \mathbf{B}_i = d\mathbf{B}_i + f^k_{ji} \mathbf{c}^j \mathbf{B}_k - \mathbf{b}_i, \quad (2.4c)$$

$$\delta_W \mathbf{C}^i = d\mathbf{C}^i - f^i_{jk} \mathbf{c}^j \mathbf{C}^k, \quad (2.4d)$$

where  $\delta_W = (S_W, \cdot)_W$  [31].

From (2.3), it follows that the Weil sigma model action is BV invariant

$$\delta_W S_W = 0. \quad (2.5)$$

Again from (2.3), it follows that the Weil sigma model BV variation operator  $\delta_W$  is nilpotent

$$\delta_W^2 = 0, \quad (2.6)$$

as can be directly verified from (2.4).

**Relation to the Weil algebra complex.** The Weil sigma model owes its name to its relation to the Weil algebra complex of  $\mathfrak{g}$ , as we shall review next (see for instance [38, 39] for background material). To any Lie algebra  $\mathfrak{g}$ , there is canonically associated the Weil algebra  $W(\mathfrak{g}) = \wedge^* \mathfrak{g}^\vee[1] \otimes \vee^* \mathfrak{g}^\vee[2]$ , the tensor product of the antisymmetric and symmetric algebras of  $\mathfrak{g}^\vee$  in degree 1 and 2, respectively. The natural  $\mathfrak{g}$ -valued generators  $\omega$ ,  $\Omega$  of  $W(\mathfrak{g})$  carry degrees 1, 2, respectively. The Weil operator  $d_W$  is the degree +1 derivation on  $W(\mathfrak{g})$  defined by

$$d_W \omega^i = \Omega^i - \frac{1}{2} f^i_{jk} \omega^j \omega^k, \quad (2.7a)$$

$$d_W \Omega^i = -f^i_{jk} \omega^j \Omega^k. \quad (2.7b)$$

It is simple to check that  $d_W$  is nilpotent

$$d_W^2 = 0. \quad (2.8)$$

Therefore,  $(W(\mathfrak{g}), d_W)$  is a differential complex. Its cohomology  $H^*(W(\mathfrak{g}), d_W)$  is actually trivial. However, it is possible to define also a  $\mathfrak{g}$  basic cohomology  $H_{\text{basic}}^*(W(\mathfrak{g}), d_W)$ , which turns out to be non trivial, as follows. One defines degree  $-1$  graded derivations  $i_i$  and degree 0 graded derivations  $l_i$  on  $W(\mathfrak{g})$  by

$$i_i \omega^j = \delta_i^j, \tag{2.9a}$$

$$i_i \Omega^j = 0, \tag{2.9b}$$

$$l_i \omega^j = -f^j_{ik} \omega^k, \tag{2.9c}$$

$$l_i \Omega^j = -f^j_{ik} \Omega^k. \tag{2.9d}$$

The derivations  $i_i$ ,  $l_i$  and  $d_W$  have the same formal properties as the contraction  $i_v$ , Lie derivative  $l_v$ , with  $v$  a vector field, and de Rham differential  $d_X$  on the graded algebra differential forms  $\Omega^*(X)$  on a manifold  $X$ . The basic subalgebra  $W(\mathfrak{g})_{\text{basic}}$  of  $W(\mathfrak{g})$  consists of those elements  $w \in W(\mathfrak{g})$  such that

$$i_i w = 0, \tag{2.10a}$$

$$l_i w = 0. \tag{2.10b}$$

$(W(\mathfrak{g})_{\text{basic}}, d_W)$  is a subcomplex of the differential complex  $(W(\mathfrak{g}), d_W)$ . Its cohomology  $H^*(W(\mathfrak{g})_{\text{basic}}, d_W)$  is by definition the basic cohomology  $H_{\text{basic}}^*(W(\mathfrak{g}), d_W)$ . It can be shown that  $W(\mathfrak{g})_{\text{basic}} = B\mathfrak{g} = \vee^* \mathfrak{g}^{\vee}[2]^G$  is the  $G$ -invariant subalgebra of the symmetric algebra  $\vee^* \mathfrak{g}^{\vee}[2] \subset W(\mathfrak{g})$  of  $\mathfrak{g}^{\vee}$  in degree 2. Actually, one has  $H_{\text{basic}}^*(W(\mathfrak{g}), d_W) \simeq B\mathfrak{g}$ , since the restriction of  $d_W$  to  $B\mathfrak{g}$  vanishes.<sup>2</sup>

As shown in ref. [31], when  $Q$  is trivial, the superfields  $\mathbf{c}$ ,  $\mathbf{C}$  describe the embedding of  $T[1]\Sigma$  into the Weil algebra  $W(\mathfrak{g})$  of the Lie algebra  $\mathfrak{g}$ . For any point  $z \in T[1]\Sigma$ , the evaluation map  $e_z : \Gamma(T[1]\Sigma, W(\mathfrak{g})) \mapsto W(\mathfrak{g})$  is a chain map of the chain complexes  $(\Gamma(T[1]\Sigma, W(\mathfrak{g})), \delta'_W)$ ,  $(W(\mathfrak{g}), d_W)$ , where  $\delta'_W$  is the nilpotent mod  $\mathbf{d}$  reduction of  $\delta_W$  obtained by setting  $\mathbf{d}\mathbf{c}^i = 0$ ,  $\mathbf{d}\mathbf{C}^i = 0$  in (2.4b), (2.4d). When  $Q$  is not trivial,  $\mathbf{c}$ ,  $\mathbf{C}$  become sections of a vector bundle of Weil algebras. The above geometrical picture still holds but only locally on  $T[1]\Sigma$ . This justifies the name given to the sigma model described here.

We shall not attempt an exhaustive study of the BV cohomology of the Weil sigma model. We shall only stress that it contains a sector isomorphic to the Weil algebra basic cohomology  $H_{\text{basic}}^*(W(\mathfrak{g}), d_W)$ . If one wished to construct a superfield out of a generic element  $w \in W(\mathfrak{g})$ , one would try with something like

$$\mathbf{w} = \sum_{p,q} \frac{1}{p!q!} w_{i_1 \dots i_p j_1 \dots j_q} \mathbf{c}^{i_1} \dots \mathbf{c}^{i_p} \mathbf{C}^{j_1} \dots \mathbf{C}^{j_q}. \tag{2.11}$$

$\mathbf{w}$ , however, is only locally defined, since the superfields  $\mathbf{c}^i$ ,  $\mathbf{C}^i$  are. To make  $\mathbf{w}$  globally defined, two requirements must be fulfilled. First, the right hand side of (2.11) must contain no occurrences of  $\mathbf{c}^i$ , since this is a generalized connection and, so, it is defined locally up

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<sup>2</sup>As is well known, the importance of the Weil algebra basic cohomology stems from its being isomorphic to the cohomology of the classifying space  $BG$  of  $G$ .

to a local gauge transformation. Second the Weil algebra element  $w$  must be  $G$ -invariant. These requirements amount to requiring  $w \in B\mathfrak{g}$ . So, we are led to consider superfields

$$\mathbf{w} = \sum_q \frac{1}{q!} w_{j_1 \dots j_q} C^{j_1} \dots C^{j_q}, \quad (2.12)$$

with  $w \in B\mathfrak{g}$ . By a simple calculation, one finds

$$\delta_W \mathbf{w} = d\mathbf{w}. \quad (2.13)$$

Hence,  $\mathbf{w}$  is a cocycle of the mod  $d$  BV cohomology. Since  $H_{\text{basic}}^*(W(\mathfrak{g}), d_W) \simeq B\mathfrak{g}$ , the mapping  $w \mapsto \mathbf{w}$  defines an isomorphism of  $H_{\text{basic}}^*(W(\mathfrak{g}), d_W)$  and a certain sector of the mod  $d$  BV cohomology.

In field theory, one is interested in the BV cohomology rather than the mod  $d$  BV cohomology, since the BV cocycles are the observables of the field theory. For any supercycle  $\mathcal{C}$  of  $T[1]\Sigma$

$$\mathbf{w}(\mathcal{C}) = \oint_{\mathcal{C}} \mathbf{w} \quad (2.14)$$

is a cocycle of the BV cohomology

$$\delta_W \mathbf{w}(\mathcal{C}) = 0. \quad (2.15)$$

For a fixed homology class  $[\mathcal{C}]$  of  $T[1]\Sigma$ , the mapping  $w \mapsto \mathbf{w}(\mathcal{C})$  defines a generally non injective homomorphism of  $H_{\text{basic}}^*(W(\mathfrak{g}), d_W)$  into a certain sector of the BV cohomology.

**The Weil sigma model in components.** One can expand the Weil sigma model fields in homogeneous components

$$\mathbf{b}_i(\mathbf{z}) = b_i(z) + \vartheta^\alpha A^+_{\alpha i}(z) + \frac{1}{2} \vartheta^\alpha \vartheta^\beta c^+_{\alpha\beta i}(z), \quad (2.16a)$$

$$\mathbf{c}^i(\mathbf{z}) = c^i(z) - \vartheta^\alpha A_\alpha{}^i(z) - \frac{1}{2} \vartheta^\alpha \vartheta^\beta b^+_{\alpha\beta}{}^i(z), \quad (2.16b)$$

$$\mathbf{B}_i(\mathbf{z}) = B_i(z) + \vartheta^\alpha \psi^+_{\alpha i}(z) + \frac{1}{2} \vartheta^\alpha \vartheta^\beta C^+_{\alpha\beta i}(z), \quad (2.16c)$$

$$\mathbf{C}^i(\mathbf{z}) = C^i(z) - \vartheta^\alpha \psi_\alpha{}^i(z) - \frac{1}{2} \vartheta^\alpha \vartheta^\beta B^+_{\alpha\beta}{}^i(z), \quad (2.16d)$$

where  $\mathbf{z} \simeq (z, \vartheta)$ ,  $z^\alpha$ ,  $\vartheta^\alpha$  being base and fiber coordinates of  $T[1]\Sigma$ . The ghost number of the various component fields is given by the degree of the superfield they appear in minus the number of  $\vartheta^\alpha$  they are multiplied by, as  $\text{deg } \vartheta^\alpha = 1$ . All component fields belong to either  $\Omega^*(\Sigma, \text{Ad } Q[n])$  or  $\Omega^*(\Sigma, \text{Ad}^\vee Q[n])$  for some  $n$  except for  $A$  which is an ordinary connection of  $Q$ . The choice of the signs of the component fields is conventional.

The action and the BV variations of the Weil sigma model (cf. eqs. (2.2), (2.4)) can be written down explicitly in terms of the components fields. The resulting expression are lengthy and are collected in appendix A for convenience.

It is interesting to study the classical version of the Weil model and compare it with known models. The classical Weil sigma model is obtained by truncating the field content



of the full Weil sigma model to the ghost number 0 sector. The classical action of the model is found to be

$$S_{Wc} = \int_{\Sigma} \left[ -b_i (F_A^i + B^{+i}) \right]. \quad (2.17)$$

This is essentially a  $BF$  like field theory. The symmetry variations of classical Weil sigma model are obtained from the BV variations of the full Weil sigma model by retaining only the ghost fields of ghost number 1,

$$\delta_{Wc} A^i = \psi^i - D_A c^i, \quad (2.18a)$$

$$\delta_{Wc} b_i = f^k_{ji} c^j b_k \quad (2.18b)$$

$$\delta_{Wc} B^{+i} = D_A \psi^i - f^i_{jk} c^j B^{+k} \quad (2.18c)$$

$$\delta_{Wc} c^i = -\frac{1}{2} f^i_{jk} c^j c^k, \quad (2.18d)$$

$$\delta_{Wc} \psi^i = -f^i_{jk} c^j \psi^k. \quad (2.18e)$$

It is simple to verify that  $S_{Wc}$  is invariant under the above field variations

$$\delta_{Wc} S_{Wc} = 0. \quad (2.19)$$

The classical field variation operator  $\delta_{Wc}$  is nilpotent

$$\delta_{Wc}^2 = 0. \quad (2.20)$$

We stress that this relation holds off-shell.

### 3. The gauge fixing of the Weil model

To yield a field theory suitable for quantization, the Weil sigma model has to be gauge fixed. To this end, we introduce two trivial pairs of fields and their antifields.

1.  $\tilde{c} \in \Omega^0(\Sigma, \text{Ad}^\vee Q[-1])$ ,  $\gamma \in \Omega^0(\Sigma, \text{Ad}^\vee Q[0])$  and their antifields  $\tilde{c}^+ \in \Omega^2(\Sigma, \text{Ad} Q[0])$ ,  $\gamma^+ \in \Omega^2(\Sigma, \text{Ad} Q[-1])$ .
2.  $\tilde{C} \in \Omega^0(\Sigma, \text{Ad}^\vee Q[-2])$ ,  $\Gamma \in \Omega^0(\Sigma, \text{Ad}^\vee Q[-1])$  and their antifields  $\tilde{C}^+ \in \Omega^2(\Sigma, \text{Ad} Q[1])$ ,  $\Gamma^+ \in \Omega^2(\Sigma, \text{Ad} Q[0])$ .

The Weil sigma model auxiliary BV odd symplectic form is

$$\Omega_{W_{\text{aux}}} = \int_{\Sigma} \left[ \delta \tilde{c}^{+i} \delta \tilde{c}_i + \delta \gamma^{+i} \delta \gamma_i + \delta \tilde{C}^{+i} \delta \tilde{C}_i + \delta \Gamma^{+i} \delta \Gamma_i \right]. \quad (3.1)$$

The Weil sigma model auxiliary BV action is

$$S_{W_{\text{aux}}} = \int_{\Sigma} \left[ \tilde{c}^{+i} \gamma_i + \tilde{C}^{+i} \Gamma_i \right]. \quad (3.2)$$

The BV variations of the auxiliary fields are

$$\delta_{W_{\text{aux}}} \tilde{c}_i = \gamma_i, \quad (3.3a)$$

$$\delta_{W_{\text{aux}}} \gamma_i = 0, \quad (3.3b)$$

$$\delta_{W_{\text{aux}}}\gamma^{+i} = -\tilde{c}^{+i}, \quad (3.3c)$$

$$\delta_{W_{\text{aux}}}\tilde{c}^{+i} = 0, \quad (3.3d)$$

$$\delta_{W_{\text{aux}}}\tilde{C}_i = -\Gamma_i, \quad (3.3e)$$

$$\delta_{W_{\text{aux}}}\Gamma_i = 0, \quad (3.3f)$$

$$\delta_{W_{\text{aux}}}\Gamma^{+i} = -\tilde{C}^{+i}, \quad (3.3g)$$

$$\delta_{W_{\text{aux}}}\tilde{C}^{+i} = 0. \quad (3.3h)$$

One has as usual

$$\delta_{W_{\text{aux}}}S_{W_{\text{aux}}} = 0. \quad (3.4)$$

$\delta_{W_{\text{aux}}}$  is nilpotent,

$$\delta_{W_{\text{aux}}}^2 = 0. \quad (3.5)$$

The gauge fixing is implemented by adding the auxiliary fields to the field content of the Weil sigma model and by adding the auxiliary field action  $S_{W_{\text{aux}}}$  to the Weil sigma model action  $S_W$ :

$$S_{W_{\text{ext}}} = S_W + S_{W_{\text{aux}}}. \quad (3.6)$$

The gauge fixed action  $I_W$  is obtained by restricting  $S_{W_{\text{ext}}}$  to a suitable Lagrangian submanifold  $\mathfrak{L}_W$  in field space

$$I_W = S_{W_{\text{ext}}}|_{\mathfrak{L}_W}. \quad (3.7)$$

$I_W$  is invariant under a BRST symmetry  $s_W$ , which is the residual BV symmetry left intact by the gauge fixing.

The Lagrangian submanifold  $\mathfrak{L}_W$  is defined in terms of a ghost number  $-1$  gauge fermion  $\Psi_W$  in the form  $\phi^+ = \delta\Psi_W/\delta\phi$ , where  $\phi$  is any field. The gauge fermion we choose has the following form:

$$\Psi_W = \int_{\Sigma} \left[ -h^{ij}b_i B_j * 1 + \tilde{C}_i D_A * \psi^i + \tilde{c}_i D_{A_0} * (A^i - A_0^i) \right], \quad (3.8)$$

where  $h$  is an Ad invariant metric on  $\mathfrak{g}$ . Above,  $*$  denotes the Hodge operator associated with a metric of  $\Sigma$ .  $A_0$  is a background connection of  $Q$ . The insertion of  $A_0$  is required by the global definedness on  $\Sigma$  of the integrand in the right hand side of (3.8). Then,  $\mathfrak{L}_W$  turns out to be explicitly defined by the constraints

$$b^{+i} = -h^{ij}B_j * 1, \quad (3.9a)$$

$$B^{+i} = -h^{ij}b_j * 1, \quad (3.9b)$$

$$c^+_i = 0, \quad (3.9c)$$

$$C^+_i = 0, \quad (3.9d)$$

$$A^+_i = *D_{A_0}\tilde{c}_i + f^k_{ji} * \psi^j \tilde{C}_k, \quad (3.9e)$$

$$\psi^+_i = - * D_A \tilde{C}_i, \quad (3.9f)$$

$$\tilde{C}^{+i} = D_A * \psi^i, \quad (3.9g)$$

$$\Gamma^{+i} = 0, \quad (3.9h)$$

$$\tilde{c}^{+i} = D_{A_0} * (A^i - A_0^i), \quad (3.9i)$$

$$\gamma^{+i} = 0. \quad (3.9j)$$

Substituting (3.9) into (A.2) in accordance with (3.7), one then finds that the gauge fixed action  $I_W$  is

$$\begin{aligned} I_W = \int_{\Sigma} \left[ \gamma_i D_{A_0} * (A^i - A_0^i) + \tilde{c}_i D_{A_0} * \psi^i - D_{A_0} \tilde{c}_i * D_A c^i + h^{ij} b_i b_j * 1 \right. \\ \left. - b_i F_A^i + B_i D_A \psi^i - D_A \tilde{C}_i * D_A C^i - (\Gamma_i + f^j_{ki} \tilde{C}_j c^k) D_A * \psi^i \right. \\ \left. + f^i_{jk} \tilde{C}_i \psi^j * \psi^k + f^{ij}_k B_i B_j C^k * 1 \right]. \end{aligned} \quad (3.10)$$

The BRST variations of the fields are obtained from (A.5), (3.3) upon restriction to  $\mathfrak{L}_W$ . They read

$$s_W A^i = \psi^i - D_A c^i, \quad (3.11a)$$

$$s_W \psi^i = -D_A C^i - f^i_{jk} c^j \psi^k, \quad (3.11b)$$

$$s_W b_i = f^k_{ji} c^j b_k + f^k_{ji} C^j B_k, \quad (3.11c)$$

$$s_W B_i = -b_i + f^k_{ji} c^j B_k, \quad (3.11d)$$

$$s_W c^i = C^i - \frac{1}{2} f^i_{jk} c^j c^k, \quad (3.11e)$$

$$s_W C^i = -f^i_{jk} c^j C^k, \quad (3.11f)$$

$$s_W \tilde{c}^i = \gamma^i, \quad (3.11g)$$

$$s_W \gamma^i = 0, \quad (3.11h)$$

$$s_W \tilde{C}^i = -\Gamma^i, \quad (3.11i)$$

$$s_W \Gamma^i = 0. \quad (3.11j)$$

One can verify directly that

$$s_W I_W = 0. \quad (3.12)$$

Further, one has

$$s_W^2 = 0. \quad (3.13)$$

In general, the BRST variation operator is nilpotent only on-shell. In this case however, it does square to 0 off-shell.

Using (3.11), it can be verified that

$$I_W = S_{W_{\text{top}}} + s_W \Psi_W, \quad (3.14)$$

where the topological action  $S_{W_{\text{top}}}$  is given by

$$S_{W_{\text{top}}} = \int_{\Sigma} \left[ -b_i F_A^i + B_i D_A \psi^i \right]. \quad (3.15)$$

This relation shows the topological nature of the theory. All dependence on the metric of  $\Sigma$  and the background connection  $A_0$  is buried inside the gauge fermion  $\Psi_W$ . The topological quantum field correlators therefore are going to be independent from these data.

The topological field theory, which we are dealing with, is in fact the 2-dimensional version of Donaldson-Witten theory [36, 37], which describes the moduli space of flat connection of a trivial principal  $G$ -bundle  $Q$ . This is easily seen from the BRST variations (3.11) obtained above. It is known that a topological field theory localizes on the BRST invariant purely bosonic on-shell configurations. Setting all the fermionic fields to zero in the BRST variations and imposing that the resulting expressions vanish leads to the equation  $b_i = 0$ , which, on shell, is equivalent to

$$F^i = 0. \tag{3.16}$$

We remark that the above procedure yields at once the full topological field theory action and the Faddeev-Popov gauge fixing action. The latter consists of those terms in the right hand side of (3.10), which depend explicitly on the background connection  $A_0$ .

#### 4. The Poisson-Weil sigma model

The Poisson-Weil sigma model stems from coupling the Weil sigma model described in section 2 and the Poisson sigma model [32, 33]. This procedure is in fact a way of gauging the latter and generalizes our original construction in [31]. As for the Weil sigma model, the covariance of the superfields of the version of the Poisson-Weil sigma model illustrated below is more general than that originally envisaged in [31], but its BV formulation is essentially the same. The reader is therefore invited again to read that paper for more details on the BV formalism used and the derivation of the classical master equation and BV variations below.

We consider a geometrical setting consisting of the following elements.

1. A closed surface  $\Sigma$ .
2. A compact connected Lie group  $G$  with Lie algebra  $\mathfrak{g}$ .
3. A principal  $G$ -bundle  $Q$  over  $\Sigma$ .
4. A manifold  $M$  carrying a smooth effective left  $G$ -action with fundamental vector field  $u \in C^\infty(M, TM \otimes \mathfrak{g}^\vee)$ .
5. A  $G$ -invariant 2-vector  $P \in C^\infty(M, \wedge^2 TM)$  and a  $G$ -equivariant  $\mathfrak{g}^\vee$ -valued scalar  $\mu \in C^\infty(M, \mathfrak{g}^\vee)$ .

The geometry associated with these data is rich and intricate. Some of its features have already emerged in the work [18–20]. Here we shall limit ourselves to indicate the aspects of it which are most directly relevant in our analysis.

The first three geometrical data are the ones entering in the definition of the Weil sigma model as illustrated in section 2. The fourth geometrical datum allows one to define

the bundle  $E_M = Q \times_G M$  with base  $\Sigma$ .  $E_M$  can be described as follows. Let  $\{U_A\}$  be a sufficiently fine open covering of  $\Sigma$ . Then, locally, one has  $E_M|_{U_A} \simeq U_A \times M$ .  $E_M$  is obtained by identifying  $(z, m_A) \in U_A \times M$  and  $(z, m_B) \in U_B \times M$  with  $z \in U_A \cap U_B \neq \emptyset$  and  $m_A = g_{AB}(z)(m_B)$ , where  $\{g_{AB}\}$  is the  $G$ -valued 1-cocycle representing  $Q$  with respect to  $\{U_A\}$  and  $g(m)$  denotes the action of the group element  $g \in G$  on the point  $m \in M$ . When  $Q$  is trivial, one has  $E_M \simeq \Sigma \times M$ . Sections  $x \in \Gamma(\Sigma, E_M)$  generalize the customary embeddings  $x : \Sigma \rightarrow M$ , which they reduce to when  $Q$  is trivial.

The bundle projection  $a_1 : T[1]\Sigma \rightarrow \Sigma$  introduced in section 2 allows one to pull-back  $E_M$  to  $T[1]\Sigma$  yielding the bundle  $a_1^*E_M$  with base space  $T[1]\Sigma$ . In terms of a fine open covering  $\{U_A\}$  of  $\Sigma$ , one has  $a_1^*E_M|_{a_1^{-1}(U_A)} \simeq a_1^{-1}(U_A) \times M$ .  $a_1^*E_M$  is obtained by identifying  $(z, m_A) \in a_1^{-1}(U_A) \times M$  and  $(z, m_B) \in a_1^{-1}(U_B) \times M$  with  $a_1(z) \in U_A \cap U_B \neq \emptyset$  and  $m_A = \mathbf{g}_{AB}(z)(m_B)$ , where  $\{\mathbf{g}_{AB}\}$  is the  $G$ -valued 1-cocycle representing  $a_1^*Q$  with respect to  $\{a_1^{-1}(U_A)\}$  defined in section 2. When  $Q$  is trivial, one has  $a_1^*E_M \simeq T[1]\Sigma \times M$ . Sections  $\mathbf{x} \in \Gamma(T[1]\Sigma, a_1^*E_M)$  generalize the customary superembeddings  $x : T[1]\Sigma \rightarrow M$ , which they reduce to when  $Q$  is trivial.

Associated with  $E_M$  are the vector bundle  $\text{Vert}TE_M$  and its dual  $\text{Vert}^*TE_M$  with base space  $E_M$ , where  $\text{Vert}TE_M = \ker \pi_{E_M^*}$ ,  $\pi_{E_M} : E_M \rightarrow \Sigma$  being the bundle projection and  $\pi_{E_M^*} : TE_M \rightarrow T\Sigma$  its tangent map. Given a fine enough open covering  $\{U_A\}$  of  $\Sigma$ , the transition functions of the bundle  $\text{Vert}TE_M$  with respect to the open covering  $\{\pi_{E_M}^{-1}(U_A)\}$  are of the form  $t_{AB}(e) = g_{AB}(z)_*(m_B)$  for  $\pi_{E_M}(e) \in U_A \cap U_B$ , where  $e \simeq (z, m_B)$  in the trivialization  $E_M|_{U_B} \simeq U_B \times M$  and  $g_*(m) : T_mM \rightarrow T_{g(m)}M$  is the tangent map at  $m \in M$  of the action  $g : M \rightarrow M$  of  $g \in G$ . Given  $x \in \Gamma(\Sigma, E_M)$ , one can define the pull-back bundles  $x^*\text{Vert}TE_M$  and  $x^*\text{Vert}^*TE_M$ , which are vector bundles with base space  $\Sigma$ . The transition functions of the bundle  $x^*\text{Vert}TE_M$  are  $t_{AB}(z) = g_{AB}(z)_*(x_B(z))$ , where  $x(z) \simeq (z, x_B(z))$  in the trivialization  $E_M|_{U_B} \simeq U_B \times M$ .

This construction can be extended by replacing  $E_M$  by  $a_1^*E_M$  and  $\Sigma$  by  $T[1]\Sigma$  throughout above. In this way, one builds the vector bundles  $\text{Vert}Ta_1^*E_M$  and its dual  $\text{Vert}^*Ta_1^*E_M$  with base  $a_1^*E_M$ . The transition function of  $\text{Vert}Ta_1^*E_M$  are of the form  $\mathbf{t}_{AB}(\mathbf{e}) = \mathbf{g}_{AB}(\mathbf{z})_*(m_B)$  for  $\pi_{a_1^*E_M}(\mathbf{e}) \in a_1^{-1}(U_A) \cap a_1^{-1}(U_B)$ , where  $\mathbf{e} \simeq (z, m_B)$  in the trivialization  $a_1^*E_M|_{a_1^{-1}(U_B)} \simeq a_1^{-1}(U_B) \times M$ . Given  $\mathbf{x} \in \Gamma(T[1]\Sigma, a_1^*E_M)$ , one can build the pull-back bundles  $\mathbf{x}^*\text{Vert}Ta_1^*E_M$  and  $\mathbf{x}^*\text{Vert}^*Ta_1^*E_M$ , which are vector bundles with base space  $T[1]\Sigma$ . The transition functions of the bundle  $\mathbf{x}^*\text{Vert}Ta_1^*E_M$  are  $\mathbf{t}_{AB}(\mathbf{z}) = \mathbf{g}_{AB}(\mathbf{z})_*(\mathbf{x}_B(\mathbf{z}))$ , where  $\mathbf{x}(\mathbf{z}) \simeq (z, \mathbf{x}_B(\mathbf{z}))$  in the trivialization  $a_1^*E_M|_{a_1^{-1}(U_B)} \simeq a_1^{-1}(U_B) \times M$ .

The fundamental vector field  $u$  satisfies the basic equivariance relation<sup>3</sup>

$$[u_i, u_j]^a = u_i^b \partial_b u_j^a - u_j^b \partial_b u_i^a = f^k{}_{ij} u_k^a. \tag{4.1}$$

The  $G$ -invariance of the 2-vector  $P$  and the  $G$ -equivariance of the scalar  $\mu$  are crucial in

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<sup>3</sup>This relation fixes also the overall sign convention for  $u$  used in the paper. According to this,  $g^a(m) = m^a - \xi^i u_i^a(m) + O(\xi^2)$  for  $g = \exp(\xi) \in G$  with  $\xi \in \mathfrak{g}$ . See appendix C.

our construction. Infinitesimally, they are equivalent to the relations

$$l_{u_i} P^{ab} = u_i^c \partial_c P^{ab} - \partial_c u_i^a P^{cb} - \partial_c u_i^b P^{ac} = 0. \quad (4.2a)$$

$$l_{u_i} \mu_j = u_i^b \partial_b \mu_j = f^k_{ij} \mu_k, \quad (4.2b)$$

The field content of the Poisson-Weil sigma model consists the following superfields.

1. The superfields of the Weil sigma model.
2. A section  $\mathbf{x} \in \Gamma(T[1]\Sigma, a_1^* E_M)$ .
3. A section  $\mathbf{y} \in \Gamma(T[1]\Sigma, \mathbf{x}^* \text{Vert}^* T a_1^* E_M[1])$ .

The BV odd symplectic form is given by

$$\Omega_{\text{PW}} = \Omega_W + \int_{T[1]\Sigma} \varrho \delta \mathbf{x}^a \delta \mathbf{y}_a, \quad (4.3)$$

where  $\Omega_W$  is the BV odd symplectic form of the Weil sigma model given in (2.1).

The action of the Poisson-Weil sigma model is

$$S_{\text{PW}} = S_W + \int_{T[1]\Sigma} \varrho \left[ \mathbf{y}_a (\mathbf{d}\mathbf{x}^a + u_i^a(\mathbf{x}) \mathbf{c}^i) + \mu_i(\mathbf{x}) \mathbf{C}^i + \frac{1}{2} P^{ab}(\mathbf{x}) \mathbf{y}_a \mathbf{y}_b \right], \quad (4.4)$$

where  $S_W$  is the action of the Weil sigma model given in (2.2). The  $G$ -invariance of  $P$  and the  $G$ -equivariance of  $\mu$  ensure the global definedness of the integrand in the right hand side of (4.4).

It can be verified by explicit computation that  $S_{\text{PW}}$  satisfies the classical master equation

$$(S_{\text{PW}}, S_{\text{PW}})_{\text{PW}} = 0, \quad (4.5)$$

where  $(\cdot, \cdot)_{\text{PW}}$  are the BV antibrackets associated with the BV form  $\Omega_{\text{PW}}$ , provided  $u, \mu, P$  satisfy the conditions

$$P^{ad} \partial_d P^{bc} + P^{bd} \partial_d P^{ca} + P^{cd} \partial_d P^{ab} = 0, \quad (4.6a)$$

$$u_i^a + P^{ab} \partial_b \mu_i = 0, \quad (4.6b)$$

[31]. Eqs. (4.6a), (4.6b) imply respectively the following properties.

1.  $P$  is a Poisson 2-vector and  $M$  is thus a Poisson manifold.
2.  $\mu$  is a moment map for the  $G$ -action, which is therefore Hamiltonian.

The BV variations of the Poisson-Weil sigma model fields are

$$\delta_{\text{PW}} \mathbf{b}_i = \delta_W \mathbf{b}_i - u_i^a(\mathbf{x}) \mathbf{y}_a, \quad (4.7a)$$

$$\delta_{\text{PW}} \mathbf{c}^i = \delta_W \mathbf{c}^i, \quad (4.7b)$$

$$\delta_{\text{PW}} \mathbf{B}_i = \delta_W \mathbf{B}_i - \mu_i(\mathbf{x}), \quad (4.7c)$$

$$\delta_{\text{PW}} \mathbf{C}^i = \delta_W \mathbf{C}^i, \quad (4.7d)$$

$$\delta_{\text{PW}}\mathbf{x}^a = d\mathbf{x}^a + u_i^a(\mathbf{x})\mathbf{c}^i + P^{ab}(\mathbf{x})\mathbf{y}_b, \quad (4.7e)$$

$$\delta_{\text{PW}}\mathbf{y}_a = d\mathbf{y}_a + \partial_a u_i^b(\mathbf{x})\mathbf{y}_b\mathbf{c}^i + \partial_a \mu_i(\mathbf{x})\mathbf{C}^i + \frac{1}{2}\partial_a P^{bc}(\mathbf{x})\mathbf{y}_b\mathbf{y}_c, \quad (4.7f)$$

where  $\delta_{\text{PW}} = (S_{\text{PW}}, \cdot)_{\text{PW}}$  and the Weil sigma model  $\delta_W$  variations are given by (2.4) [31].

From (4.5), it follows that the Poisson-Weil sigma model action is BV invariant

$$\delta_{\text{PW}}S_{\text{PW}} = 0. \quad (4.8)$$

Again from (4.5), it follows that the Poisson-Weil sigma model BV variation operator  $\delta_{\text{PW}}$  is nilpotent

$$\delta_{\text{PW}}^2 = 0. \quad (4.9)$$

**Relation to the Hamiltonian basic Poisson-Lichnerowicz cohomology.** When conditions (4.6), (4.2) are satisfied, if  $a \in \mathfrak{g}^\vee$  with coadjoint orbit  $\mathcal{O}_a$  and  $\mu^{-1}(\mathcal{O}_a)$  is a submanifold of  $M$  on which  $G$  acts freely and properly, then the quotient  $M_a = \mu^{-1}(\mathcal{O}_a)/G$  inherits a Poisson structure  $P_a$ , by a classic result of Marsden and Ratiu [40]. One considers mainly  $M_0 = \mu^{-1}(\{0\})/G \equiv M//G$ .

From the above discussion, it appears that the Poisson-Weil sigma model on a Poisson manifold  $M$  carrying a Hamiltonian action of a group  $G$  encodes the Hamiltonian reduction of  $M$  by  $G$ . Upon suitably restricting the image of  $\mathbf{x}$  to  $\mu^{-1}(\mathcal{O}_a)$ , one expects to obtain some kind of sigma model on  $M_a$ . When the principal bundle  $Q$  is trivial, this should be an ordinary Poisson sigma model on  $M_a$ . The embedding fields of the model are then just maps  $x : \Sigma \rightarrow M_a$ . Conversely, when  $Q$  is non trivial, one should obtain a generalized Poisson sigma model on  $M_a$ . The embedding fields of the model are then sections  $x \in \Gamma(\Sigma, E_M)$  such that, in any trivialization  $E_M|_{U_A} \simeq U_A \times M$ ,  $x_A(z) \in \mu^{-1}(\mathcal{O}_a)$  for  $z \in U_A$ . (Note that this property is independent from the chosen trivialization). Intuitively, they are some kind of “ $Q$ -twisted” maps  $x : \Sigma \rightarrow M_a$ . These facts should be reflected in the BV cohomology of the Poisson-Weil sigma model, which we explore next. As we shall see, this investigation will bring us close to the boundary of known mathematics.

Recall that a Poisson manifold  $M$  with Poisson 2-vector field  $P$  is characterized by the algebra of multivector fields  $C^\infty(M, \wedge^*TM)$  and by the Poisson-Lichnerowicz differential  $\sigma_{\text{PL}} = [P, \cdot]$ , where  $[\cdot, \cdot]$  are the Schouten brackets on  $C^\infty(M, \wedge^*TM)$  (see for instance [41] for background material). Since  $\sigma_{\text{PL}}$  is nilpotent,  $(C^\infty(M, \wedge^*TM), \sigma_{\text{PL}})$  is a differential complex, the Poisson-Lichnerowicz complex. The associated cohomology is the Poisson-Lichnerowicz cohomology  $H_{\text{PL}}^*(M)$ . Each cohomology class is represented by a Poisson-Lichnerowicz cocycle, that is a multivector field  $\alpha \in C^\infty(M, \wedge^*TM)$  satisfying

$$\sigma_{\text{PL}}\alpha = 0. \quad (4.10)$$

This cocycle is defined up to a Poisson-Lichnerowicz coboundary, i. e. a multivector field belonging to the image of  $\sigma_{\text{PL}}$ .

A simple analysis shows that  $H_{\text{PL}}^0(M)$  is the algebra of Casimir functions of  $M$  and  $H_{\text{PL}}^1(M)$  is the quotient of the space of Poisson vector fields of  $M$  over the space of Hamiltonian vector fields, etc. Further, the Poisson 2-vector field  $P$ , viewed as an element of

$C^\infty(M, \text{Hom}(T^*M, TM))$ , induces a homomorphism  $P^\#$  of the ordinary de Rham cohomology  $H_{\text{dR}}^*(M)$  into  $H_{\text{PL}}^*(M)$ , which is an isomorphism in the symplectic case.

Suppose that  $M$  carries a Hamiltonian smooth effective left  $G$ -action with fundamental vector field  $u \in C^\infty(M, TM \otimes \mathfrak{g}^\vee)$  and  $G$ -equivariant moment map  $\mu \in C^\infty(M, \mathfrak{g}^\vee)$  and leaving  $P$  invariant. We call a multivector field  $\alpha \in C^\infty(M, \wedge^*TM)$  Hamiltonian basic, if  $\alpha$  satisfies the conditions

$$i_{d\mu_i}\alpha = 0, \quad (4.11a)$$

$$l_{u_i}\alpha = 0, \quad (4.11b)$$

where  $i_\omega$  denotes contraction with the 1-form  $\omega \in C^\infty(M, T^*M)$  and  $l_v$  is the Lie derivative along the vector field  $v \in C^\infty(M, TM)$ , i.e. if  $\alpha$  is  $G$ -invariant and tangent to the  $\mu$  fibers. The terminology is justified by the analogy to the notion of basic forms of a manifold with a group action. We denote by  $C^\infty(M, \wedge^*TM)_{\text{basic}}$  the subalgebra of  $C^\infty(M, \wedge^*TM)$  spanned by the Hamiltonian basic multivector fields. Using the relations

$$i_{d\mu_i}\sigma_{\text{PL}} + \sigma_{\text{PL}}i_{d\mu_i} = l_{u_i}, \quad (4.12a)$$

$$\sigma_{\text{PL}}l_{u_i} - l_{u_i}\sigma_{\text{PL}} = 0, \quad (4.12b)$$

$$l_{u_i}i_{d\mu_i} - l_{u_i}i_{d\mu_i} = f^k_{ij}i_{d\mu_k}, \quad (4.12c)$$

one shows that  $(C^\infty(M, \wedge^*TM)_{\text{basic}}, \sigma_{\text{PL}})$  is a subcomplex of  $(C^\infty(M, \wedge^*TM), \sigma_{\text{PL}})$ , the Hamiltonian basic Poisson-Lichnerowicz complex. The associated cohomology is the Hamiltonian basic Poisson-Lichnerowicz cohomology  $H_{\text{PLbasic}}^*(M)$ . Each cohomology class is represented by a Hamiltonian basic Poisson-Lichnerowicz cocycle, i.e. a multivector field  $\alpha \in C^\infty(M, \wedge^*TM)$  satisfying the conditions (4.10), (4.11). This cocycle is defined up to a Hamiltonian basic Poisson-Lichnerowicz coboundary, i. e. a multivector field belonging to the image of  $\sigma_{\text{PL}}$  restricted to  $C^\infty(M, \wedge^*TM)_{\text{basic}}$ .

Repeating the analysis done for ordinary Poisson cohomology, one can show that  $H_{\text{PLbasic}}^0(M)$  is the algebra of ordinary Casimir functions of  $M$  and  $H_{\text{PLbasic}}^1(M)$  is the quotient of the space of  $G$ -invariant Poisson vector fields of  $M$  tangent to the  $\mu$  fibers over the space of Hamiltonian vector fields with  $G$ -invariant Hamiltonians, etc. Further,  $P$  induces a homomorphism  $P^\#$  of the ordinary basic de Rham cohomology  $H_{\text{dRbasic}}^*(M)$  into  $H_{\text{PLbasic}}^*(M)$ , which is an isomorphism in the symplectic case.

The Hamiltonian basic Poisson-Lichnerowicz cohomology  $H_{\text{PLbasic}}^*(M)$  was introduced and studied in a more general context by Ginzburg in [42]. It is natural to expect  $H_{\text{PLbasic}}^*(M)$  to be related to the Poisson-Lichnerowicz cohomology of the reduced Poisson manifolds  $M_a$  defined above. However, to the best of our knowledge, so far this relation has not been elucidated in the mathematical literature except for symplectic manifolds in [43] by Kirwan, who showed the existence of a natural surjective generally non injective homomorphism  $\kappa : H_{\text{dRbasic}}^*(M) \simeq H_{\text{PLbasic}}^*(M) \rightarrow H_{\text{dR}}^*(M_0)$ . Virtually nothing is known for more general Poisson manifolds.

We shall not attempt an exhaustive study of the BV cohomology of the Poisson-Weil sigma model. We shall only try to highlight some of its novel features and its relation to the Hamiltonian basic Poisson-Lichnerowicz cohomology. If one wished to construct a



superfield out of a generic multivector field  $\alpha \in C^\infty(M, \wedge^* TM)$ , one would start by trying with something of the form

$$\alpha = \sum_p \frac{1}{p!} \alpha^{a_1 \dots a_p}(\mathbf{x}) \mathbf{y}_{a_1} \cdots \mathbf{y}_{a_p}. \quad (4.13)$$

This object is however only locally defined since the superfields  $\mathbf{x}^a$ ,  $\mathbf{y}_a$  are defined only up to a local  $G$ -action. To render  $\alpha$  globally defined, one has to demand the multivector field  $\alpha$  to be  $G$ -invariant,  $\alpha \in C^\infty(M, \wedge^* TM)^G$ . Infinitesimally, this is equivalent to (4.11b).

A straightforward calculation yields

$$\begin{aligned} \delta_{\text{PW}} \alpha &= d\alpha + \sum_p \frac{1}{p!} (i_{d\mu_i} \alpha)^{a_1 \dots a_p}(\mathbf{x}) C^i \mathbf{y}_{a_1} \cdots \mathbf{y}_{a_p} \\ &\quad - \sum_p \frac{1}{p!} (\sigma_{\text{PL}} \alpha)^{a_1 \dots a_p}(\mathbf{x}) \mathbf{y}_{a_1} \cdots \mathbf{y}_{a_p}. \end{aligned} \quad (4.14)$$

Hence, one has

$$\delta_{\text{PW}} \alpha = d\alpha \quad (4.15)$$

provided  $\alpha \in C^\infty(M, \wedge^* TM)^G$  satisfies (4.10), (4.11a). In that case,  $\alpha$  is a cocycle of the mod  $d$  BV cohomology. Furthermore, the mapping  $\alpha \mapsto \alpha$  defines an isomorphism of  $H_{\text{PLbasic}}^*(M)$  and a distinguished sector of the mod  $d$  BV cohomology.

As already remarked earlier, in field theory, one is interested in the BV cohomology rather than the mod  $d$  BV cohomology, since BV cocycles are observables. For any super-cycle  $\mathcal{C}$  of  $T[1]\Sigma$ ,

$$\alpha(\mathcal{C}) = \oint_{\mathcal{C}} \alpha \quad (4.16)$$

is a cocycle of the BV cohomology

$$\delta_{\text{PW}} \alpha(\mathcal{C}) = 0. \quad (4.17)$$

For a fixed homology class  $[\mathcal{C}]$  of  $T[1]\Sigma$ , the mapping  $\alpha \mapsto \alpha(\mathcal{C})$  defines a generally non injective homomorphism of  $H_{\text{PLbasic}}^*(M)$  into the BV cohomology.

We conclude that, for a fixed homology class  $[\mathcal{C}]$  of  $T[1]\Sigma$ ,  $\alpha(\mathcal{C})$  is an observable provided  $\alpha$  is a Hamiltonian basic Poisson-Lichnerowicz cocycle. This establishes a homomorphism of the Hamiltonian basic Poisson-Lichnerowicz cohomology  $H_{\text{PLbasic}}^*(M)$  into the Poisson-Weil BV cohomology.

In [42], Ginzburg also defined the equivariant Poisson-Lichnerowicz cohomology  $H_{\text{PLG}}^*(M)$ . This can be realized in two different but equivalent models. In the Weil model, one relies on the Weil algebra  $(W(\mathfrak{g}), d_W)$  complex described in section 2.  $H_{\text{PLG}}^*(M)$  is the cohomology of the complex  $((C^\infty(M, \wedge^* TM) \otimes W(\mathfrak{g}))_{\text{basic}}, \sigma_{\text{PLW}})$ , where basicity is defined in terms of the graded derivations  $i_{W_i} = i_{d\mu_i} + i_i$ ,  $l_{W_i} = l_{u_i} + l_i$ , by extending (2.10), (4.11) in obvious fashion, and  $\sigma_{\text{PLW}} = \sigma_{\text{PL}} + d_W$ . In the Cartan model,  $H_{\text{PLG}}^*(M)$  is the cohomology of the complex  $((C^\infty(M, \wedge^* TM) \otimes \vee^* \mathfrak{g}^\vee[2])^G, \sigma_{\text{PLC}})$ , where  $G$ -invariance is defined in terms of  $l_{C_i} = l_{u_i} + l_i$  and  $\sigma_{\text{PLC}} = \sigma_{\text{PL}} - \Omega^k i_{d\mu_k}$ , with  $\Omega^i$  the degree 2 generators of  $\vee^* \mathfrak{g}^\vee[2]$  and  $l_i$  defined as in (2.9d).

When  $G$  is compact and  $\mu$  is a submersion onto  $\mathfrak{g}^\vee$ ,  $H_{\text{PLbasic}}^*(M)$  is isomorphic to the equivariant Poisson-Lichnerowicz cohomology  $H_{\text{PL}G}^*(M)$  [42]. Using the Cartan model for simplicity, a class of  $H_{\text{PLbasic}}^*(M)$  is represented by a  $G$ -invariant multivector field  $\alpha \in (C^\infty(M, \wedge^* TM) \otimes \vee^* \mathfrak{g}^\vee[2])^G$ ,

$$l_{C_i} \alpha = 0, \quad (4.18)$$

satisfying the cocycle condition

$$\sigma_{\text{PLC}} \alpha = 0. \quad (4.19)$$

This suggests a possible generalization of the ansatz (4.13) of the form

$$\alpha = \sum_{p,q} \frac{1}{p!q!} \alpha^{a_1 \dots a_p}_{i_1 \dots i_q}(\mathbf{x}) \mathbf{y}_{a_1} \dots \mathbf{y}_{a_p} C^{i_1} \dots C^{i_q}, \quad (4.20)$$

where  $\alpha \in (C^\infty(M, \wedge^* TM) \otimes \vee^* \mathfrak{g}^\vee[2])^G$ . The  $G$ -invariance of  $\alpha$  is required by the proper global definedness of the superfield  $\alpha$ . A straightforward calculation leads to

$$\delta_{\text{PW}} \alpha = \mathbf{d} \alpha - \sum_{p,q} \frac{1}{p!q!} (\sigma_{\text{PLC}} \alpha)^{a_1 \dots a_p}_{i_1 \dots i_q}(\mathbf{x}) \mathbf{y}_{a_1} \dots \mathbf{y}_{a_p} C^{i_1} \dots C^{i_q}.$$

Hence,  $\alpha$  satisfies (4.15), provided  $\alpha \in (C^\infty(M, \wedge^* TM) \otimes \vee^* \mathfrak{g}^\vee[2])^G$  satisfies (4.19). In this way, proceeding exactly in the same way as above, one can construct observables of the field theory. However, this procedure is not going to yield genuinely new observables. In fact, the inclusion  $(C^\infty(M, \wedge^* TM)_{\text{basic}}) \subset (C^\infty(M, \wedge^* TM) \otimes \vee^* \mathfrak{g}^\vee[2])^G$  induces the isomorphism  $H_{\text{PLbasic}}^*(M) \simeq H_{\text{PL}G}^*(M)$  mentioned above. This means that any mod  $\mathbf{d}$  BV cocycle of the form (4.20) is always BV cohomologous to one of the form (4.13).

**The Poisson-Weil sigma model in components.** One can expand the Poisson-Weil sigma model fields in homogeneous components. Relations (2.16) still hold. Further, one has

$$\mathbf{x}^a(z) = x^a(z) + \vartheta^\alpha \eta^+_{\alpha a}(z) - \frac{1}{2} \vartheta^\alpha \vartheta^\beta y^+_{\alpha\beta a}(z), \quad (4.21a)$$

$$\mathbf{y}_a(z) = y_a(z) + \vartheta^\alpha \eta_{\alpha a}(z) + \frac{1}{2} \vartheta^\alpha \vartheta^\beta x^+_{\alpha\beta a}(z). \quad (4.21b)$$

Again, the ghost number of the various component fields is given by the degree of the superfield they appear in minus the number of  $\vartheta^\alpha$  they are multiplied by. The covariance properties of the component fields are intricate, but they are completely determined by those of the superfield which they belong to. Again, the choice of the signs is conventional.

The action and the BV variations of the Poisson-Weil sigma model (cf. eqs. (4.4), (4.7)) can be written down explicitly in terms of the components fields. The resulting expression are rather messy and are collected in appendix B for convenience.

It is interesting to study the classical version of the Poisson-Weil model and compare it with that of the ordinary Poisson model. As for the classical Weil sigma model, the classical Poisson-Weil sigma model is obtained truncating the field content of the full Poisson-Weil

sigma model to the ghost number 0 sector. The classical action of the model is then found to be

$$S_{\text{PWc}} = S_{\text{Wc}} + \int_{\Sigma} \left[ -\mu_i(x)B^{+i} + \eta_a D_A x^a + \frac{1}{2} P^{ab}(x)\eta_a\eta_b \right], \quad (4.22)$$

where the classical Weil sigma model action  $S_{\text{Wc}}$  is given in (2.17). Again, as for the classical Weil sigma model, the symmetry variations of classical Poisson-Weil sigma model are obtained from the BV variations of the full Poisson-Weil sigma model by retaining only the ghost fields of ghost number 1,

$$\delta_{\text{PWc}} A^i = \delta_{\text{Wc}} A^i, \quad (4.23a)$$

$$\delta_{\text{PWc}} b_i = \delta_{\text{Wc}} b_i - u_i^a(x)y_a, \quad (4.23b)$$

$$\delta_{\text{PWc}} B^{+i} = \delta_{\text{Wc}} B^{+i}, \quad (4.23c)$$

$$\delta_{\text{PWc}} c^i = \delta_{\text{Wc}} c^i, \quad (4.23d)$$

$$\delta_{\text{PWc}} \psi^i = \delta_{\text{Wc}} \psi^i, \quad (4.23e)$$

$$\delta_{\text{PWc}} x^a = P^{ab}(x)y_b + u_i^a(x)c^i, \quad (4.23f)$$

$$\delta_{\text{PWc}} \eta_a = D_A y_a + \partial_a P^{bc}(x)\eta_b y_c + \partial_a u_i^b(x)\eta_b c^i - \partial_a \mu_i(x)\psi^i \quad (4.23g)$$

$$\delta_{\text{PWc}} y_a = \frac{1}{2} \partial_a P^{bc}(x)y_b y_c + \partial_a u_i^b(x)y_b c^i, \quad (4.23h)$$

where the classical Weil sigma model  $\delta_{\text{Wc}}$  variations are given by (2.18). One check that  $S_{\text{PWc}}$  is invariant under the above field variations,

$$\delta_{\text{PWc}} S_{\text{PWc}} = 0. \quad (4.24)$$

The classical field variation operator  $\delta_{\text{PWc}}$  is nilpotent but only on-shell,

$$\delta_{\text{PWc}}^2 = 0 \quad \text{on-shell.} \quad (4.25)$$

## 5. The gauge fixing of the Poisson-Weil model

In this section, we carry out the gauge fixing of the Poisson-Weil sigma model. Unlike the gauge fixing of the Weil sigma model, which is essentially unique, the gauge fixing of the Poisson-Weil sigma model can in principle be carried out in several generally inequivalent ways depending on the nature of the target space geometry. Exploring all the possibilities is out question. Below, we concentrate on a gauge fixing prescription that leads to an interesting topological field theory.

We assume that the data defining the Poisson-Weil sigma model satisfy the following additional requirements.

1. The manifold  $M$  is endowed with a Kaehler structure.
2. The  $G$ -action on  $M$  preserves the Kaehler structure.
3. The  $G$ -invariant 2-vector  $P$  is the one canonically associated with the Kaehler structure.

By a Kaehler structure, we mean a pair  $(J, g)$  formed by an almost complex structure  $J$  and a Riemannian metric  $g$ , such that  $g$  is Hermitian with respect to  $J$  and  $J$  is parallel with respect to the Levi-Civita connection of  $g$ . The almost complex structure  $J$  is then automatically integrable and, thus, a complex structure. The Kaehler form  $\omega = gJ$  defines a symplectic structure and thus a Poisson structure  $P = \omega^{-1}$ . Explicitly

$$P^{rs} = 0, \quad P^{r\bar{s}} = -ig^{r\bar{s}} \quad \text{and c. c.} \quad (5.1)$$

The  $G$ -invariance of the Kaehler structure entails the  $G$ -invariance of  $P$ .

As explained in section 4, the consistency of the model requires the  $G$ -action to be Hamiltonian with moment map  $\mu$ . Explicitly,

$$u^r{}_i = ig^{r\bar{s}}\partial_{\bar{s}}\mu_i \quad \text{and c. c.} \quad (5.2)$$

The invariance of the Kaehler structure under the  $G$ -action entails that the fundamental vector field  $u$  of the  $G$ -action is both holomorphic and Killing. This leads to the relations

$$\nabla_{\bar{r}}u^s{}_i = 0 \quad \text{and c. c.}, \quad (5.3a)$$

$$\nabla_{\bar{r}}u^{\bar{s}}{}_i + g_{\bar{r}t}g^{\bar{s}u}\nabla_u u^t{}_i = 0 \quad \text{and c. c.} \quad (5.3b)$$

Combining (5.2), (5.3a), (5.3b), one finds that  $\mu$  must satisfy the equation

$$\nabla_r\partial_s\mu_i = 0 \quad \text{and c. c.} \quad (5.4)$$

Proceeding in a way analogous to that of the Weil sigma model, the gauge fixing is implemented by adding the auxiliary fields of the Weil sigma model (cf. section 3) to the field content of the Poisson-Weil sigma model and by adding the auxiliary field action  $S_{W_{\text{aux}}}$  (cf. section 3.2) to the Poisson-Weil sigma model action  $S_{PW}$ :

$$S_{PW_{\text{ext}}} = S_{PW} + S_{W_{\text{aux}}}. \quad (5.5)$$

The gauge fixed action  $I_{PW}$  is obtained by restricting  $S_{PW_{\text{ext}}}$  to a suitable Lagrangian submanifold  $\mathfrak{L}_{PW}$  in field space,

$$I_{PW} = S_{PW_{\text{ext}}}|_{\mathfrak{L}_{PW}}. \quad (5.6)$$

$I_{PW}$  is invariant under a BRST symmetry  $s_{PW}$ , which is the residual BV symmetry left intact by the gauge fixing.

The gauge fixing requires, among other things, the choice of a metric of  $\Sigma$ . In this way, as is well-known,  $\Sigma$  acquires in canonical fashion a complex structure. The tangent bundle of  $\Sigma$  splits then in its holomorphic and antiholomorphic components  $T\Sigma = T^{(1,0)}\Sigma \oplus T^{(0,1)}\Sigma$  and similarly for the cotangent bundle. Henceforth, we conveniently redefine our notation according to  $\phi^{(1,0)} \rightarrow \phi_c, \phi^{(0,1)} \rightarrow \bar{\phi}_c$  for a given 1-form field  $\phi \in \Omega^*(\Sigma)$ .

We implement the gauge fixing, by using the gauge fixing conditions (3.9) previously employed in the Weil sector of the model and the further conditions

$$\eta_{cr} = 0 \quad \text{and c. c.}, \quad (5.7a)$$

$$\eta^+{}_c{}^r = 0 \quad \text{and c. c.}, \quad (5.7b)$$

$$x^+_r = 0 \quad \text{and c. c.}, \quad (5.7c)$$

$$y^{+r} = 0 \quad \text{and c. c.}, \quad (5.7d)$$

[29, 44]. Using (4.3), it is easy to see that these define a Lagrangian submanifold  $\mathfrak{L}_{PW}$  in field space. Note that, unlike the Weil sigma model, the condition (5.7) are not derived directly from a gauge fermion, but that does not matter as long as  $\mathfrak{L}_{PW}$  is Lagrangian as required.

After a computation, we find

$$\begin{aligned} I_{PW} = I_W + \int_{\Sigma} \left[ & ig_{\bar{r}s}(x) \bar{D}_{Ac} x^{\bar{r}} D_{Ac} x^s \quad (5.8) \\ & + \bar{\eta}^+_c{}^r (D_{\nabla Ac} y_r - \partial_r \mu_i(x) \psi_c^i) + \eta^+_c{}^{\bar{r}} (\bar{D}_{\nabla Ac} y_{\bar{r}} - \partial_{\bar{r}} \mu_i(x) \bar{\psi}_c^i) \\ & + \bar{\eta}^+_c{}^r \eta^+_c{}^{\bar{s}} (-iR^{t\bar{u}}{}_{r\bar{s}}(x) y_t y_{\bar{u}} + \partial_r \partial_{\bar{s}} \mu_i(x) C^i) \\ & + b_i h^{ij} \mu_j(x) * 1 - B_i h^{ij} (u^r{}_j(x) y_r + u^{\bar{r}}{}_j(x) y_{\bar{r}}) * 1 \right], \end{aligned}$$

where  $I_W$  is the gauge fixed Weil sigma model action (cf. eq. (3.10)) and  $D_{Ac}$ ,  $\bar{D}_{Ac}$  are the holomorphic and antiholomorphic component of the gauge covariant derivative operator  $D_A$  (cf. eqs. (A.4), (B.3), (B.5)) and we have defined

$$D_{\nabla Ac} y_r = D_{Ac} y_r - \Gamma^s{}_{tr}(x) D_{Ac} x^t y_s \quad \text{and c. c.}, \quad (5.9)$$

which is both gauge and general coordinate covariant (see appendix C.). In the above expression, wedge product of forms is understood again. Expression (5.8) is obtained upon eliminating the fields

$$\bar{\eta}'_{cr} = \bar{\eta}_{cr} - \Gamma^u{}_{tr}(x) \bar{\eta}^+_c{}^t y_u - ig_{r\bar{s}}(x) \bar{D}_{Ac} x^{\bar{s}} \quad \text{and c. c.}, \quad (5.10)$$

which decouple from all the other.

The Poisson-Weil sigma model BRST variations of the fields are obtained from (A.5), (3.3), (B.4) upon restriction to  $\mathfrak{L}_{PW}$ . They read

$$s_{PW} A^i = s_W A^i, \quad (5.11a)$$

$$s_{PW} \psi^i = s_W \psi^i, \quad (5.11b)$$

$$s_{PW} b_i = s_W b_i - u^r{}_i(x) y_r - u^{\bar{r}}{}_i(x) y_{\bar{r}}, \quad (5.11c)$$

$$s_{PW} B_i = s_W B_i - \mu_i(x) \quad (5.11d)$$

$$s_{PW} c^i = s_W c^i, \quad (5.11e)$$

$$s_{PW} C^i = s_W C^i, \quad (5.11f)$$

$$s_{PW} \tilde{c}^i = s_W \tilde{c}^i, \quad (5.11g)$$

$$s_{PW} \gamma^i = s_W \gamma^i, \quad (5.11h)$$

$$s_{PW} \tilde{C}^i = s_W \tilde{C}^i, \quad (5.11i)$$

$$s_{PW} \Gamma^i = s_W \Gamma^i, \quad (5.11j)$$

$$s_{PW}x^r = -ig^{r\bar{s}}(x)y_{\bar{s}} + u^r{}_i(x)c^i \quad \text{and c.c.}, \quad (5.11k)$$

$$s_{PW}y_r = \Gamma^s{}_{tr}(x)(-ig^{t\bar{u}}(x)y_{\bar{u}} + u^t{}_i(x)c^i)y_s + \nabla_r u^s{}_i(x)y_s c^i + \partial_r \mu_i(x)C^i \quad \text{and c.c.}, \quad (5.11l)$$

$$s_{PW}\bar{\eta}^+{}_c{}^r = -\Gamma^r{}_{ts}(x)(-ig^{t\bar{u}}(x)y_{\bar{u}} + u^t{}_i(x)c^i)\bar{\eta}^+{}_c{}^s + \bar{D}_{Ac}x^r + \nabla_s u^r{}_i(x)\bar{\eta}^+{}_c{}^s c^i \quad \text{and c.c.}, \quad (5.11m)$$

where the Weil sigma model  $s_W$  BRST variations are given by (3.11). One can verify directly that  $I_{PW}$  is BRST invariant

$$s_{PW}I_{PW} = 0. \quad (5.12)$$

Further, one has

$$s_{PW}^2 = 0 \quad \text{on shell.} \quad (5.13)$$

Unlike the Weil sigma model, the Poisson-Weil BRST variation operator is nilpotent only on-shell.

It is easy to see that the field theory we have obtained by gauge fixing is topological. One defines a ghost number  $-1$  gauge fermion  $\Psi_{PW}$  by

$$\Psi_{PW} = \Psi_W + \int_{\Sigma} \left[ \frac{1}{2}ig_{r\bar{s}}(x)\bar{\eta}^+{}_c{}^r D_{Ac}x^{\bar{s}} - \frac{1}{2}ig_{\bar{r}s}(x)\eta^+{}_c{}^{\bar{r}} \bar{D}_{Ac}x^s \right], \quad (5.14)$$

where  $\Psi_W$  is the gauge fermion of the Weil sigma model given by (3.8). Using (5.11), it can be verified that

$$I_{PW} = S_{PW\text{top}} + s_{PW}\Psi_{PW} \quad \text{on shell,} \quad (5.15)$$

where the Poisson-Weil topological action  $S_{W\text{top}}$  is given by

$$S_{PW\text{top}} = S_{W\text{top}} + \int_{\Sigma} x^*{}_A \omega, \quad (5.16)$$

with the Weil topological action  $S_{W\text{top}}$  given by (3.15). The globally defined 2-form  $x^*{}_A \omega$  is the gauge covariant pull-back of  $\omega$

$$x^*{}_A \omega = \frac{1}{2}\omega_{ab}(x)D_A x^a D_A x^b = x^* \omega + d(\mu_i(x)A^i) - \mu_i(x)F^i. \quad (5.17)$$

The above expression is obtained by using, among other things, the remarkable gauge covariant Kaehler identity

$$ig_{\bar{r}s}(x)\bar{D}_{Ac}x^{\bar{r}} D_{Ac}x^s - ig_{r\bar{s}}(x)\bar{D}_{Ac}x^r D_{Ac}x^{\bar{s}} = x^*{}_A \omega. \quad (5.18)$$

This calculation shows the topological nature of the theory. All dependence on the metric of  $\Sigma$  and the background connection  $A_0$  is again buried inside the gauge fermion  $\Psi_{PW}$ . The topological quantum field correlators, therefore, are going to be independent from these data.

The topological field theory which we are dealing with has been studied by Baptista in a series of papers [18–20]. It describes the moduli space of solutions of the so called vortex

equations [45–50]. Strictly speaking, the sigma model Lagrangian obtained above differs from Baptista’s. However, this may be simply a gauge fixing artifact. The fact that our sigma model and Baptista’s have the same field content and localize on the same space of field configurations, as we show momentarily, indicates that they are the same topological field theory. As well known, topological field theories have BRST exact Lagrangians and this makes them invariant as field theories under large classes of deformations. So, it is not surprising that the same topological field theory may have several Lagrangian realizations. What characterizes a topological field theory is its field content and the space of field configurations on which the field theory localizes. Indeed, the path integral of the field theory is just a complicated way of writing a functional Dirac delta with support on such configurations. Thus, one expects two such theories sharing the same set of fields and localizing on the same field configurations to be equivalent. However, a complete proof of this statement would require an in depth analysis of the BRST cohomologies of the two theories and a proof of their equivalence, a task which is beyond the scope of the present paper.

Let us now show that the topological field theory we have obtained describes the moduli space of solutions of the vortex equations, as claimed in the previous paragraph. To begin with, we note that the geometrical data of the gauge fixed Poisson-Weil sigma model are precisely the same as those of the vortex equations: a principal  $G$ -bundle  $Q$  over a Riemann surface  $\Sigma$  and a Kaehler manifold  $M$  with a Hamiltonian effective action preserving the Kaehler structure. The field configurations, on which our topological field theory localizes, are the BRST invariant purely bosonic on-shell configurations. They are easily obtained from the expression (5.11) of the BRST variations obtained above. Setting all the fermionic fields to zero in (5.11) and imposing that the resulting expressions vanish on shell leads to the equations

$$F^i + h^{ij}\mu_j(x) * 1 = 0, \tag{5.19}$$

$$\overline{D}_{Ac}x^r = 0, \tag{5.20}$$

which are precisely the vortex equations.

The vortex configurations are extrema of the energy functional

$$\mathcal{E} = \int_{\Sigma} \left[ \frac{1}{2}h_{ij}F^i * F^j + \frac{1}{2}g_{ab}(x)D_Ax^a * D_Ax^b + \frac{1}{2}h^{ij}\mu_i\mu_j(x) * 1 \right], \tag{5.21}$$

first written down in [45, 50]. However, they are not generic extrema. They are instanton like energy minimizing configurations. Indeed, by means of Bogomolny type manipulations, one can show that  $\mathcal{E}$  can be written as

$$\begin{aligned} \mathcal{E} = \eta_{E_M} + \int_{\Sigma} \left[ -2ig_{r\bar{s}}(x)\overline{D}_{Ac}x^r D_Ac x^{\bar{s}} \right. \\ \left. + \frac{1}{2}h_{ij}(F^i + h^{ik}\mu_k(x) * 1) * (F^j + h^{jl}\mu_l(x) * 1) \right], \end{aligned} \tag{5.22}$$

where  $\eta_{E_M}$  is given by

$$\eta_{E_M} = - \int_{\Sigma} \left[ x^* \omega + d(\mu_i(x)A^i) \right]. \tag{5.23}$$

$\eta_{E_M}$  depends only on the homotopy class of  $x$  and is independent from  $A$ . It is thus a topological invariant characterizing the bundle  $E_M$ . The remaining term in the right hand side of (5.22) is positive definite and vanishes precisely, when the vortex equations (5.19), (5.20) are satisfied. Thus, the energy  $\mathcal{E}$  is minimized by the vortex configurations and the minimum equals the topological invariant  $\eta_{E_M}$ . In this sense, vortex configurations are akin to instantons.

Our gauged topological sigma model can be viewed as a topological field theoretic completion of a purely bosonic theory with action  $\mathcal{E}$ . Indeed, the ghost number 0 sector  $I_{PW}^0$  of the action  $I_{PW}$  after algebraically eliminating the auxiliary field  $b$  is given by

$$I_{PW}^0 = -\frac{1}{2}\eta_{E_M} - \frac{1}{2}\mathcal{E}. \tag{5.24}$$

The topological sigma model, which we have obtained, is in fact the gauged version of Witten's  $A$ -model originally worked out in [11, 12]. In the case where the group  $G$  is trivial, the action  $I_{PW}$  reduces indeed to the well-known action of the  $A$ -model [44].

The  $A$ -model is known to be related to the quantum cohomology of the target manifold  $M$ : its correlators compute the Gromov-Witten invariants. The importance of the vortex equation moduli space stems from the realization that it enters the definition of the Hamiltonian Gromov-Witten invariants [46].

## 6. Outlook

The constructions expounded in this paper are likely to be extendable in several directions.

We have formulated the Weil sigma model for a principal  $G$ -bundle  $Q$  over  $\Sigma$  with  $G$  a Lie group. One possibility would be to generalize the model to the case where  $G$  is a Poisson-Lie group. One expects the Lie bialgebra structure of  $\mathfrak{g}$  to play a basic role in this case. The Weil sigma model described in the paper would be the special case where  $G$  has the trivial Poisson structure.

As a further step, one may try to couple the generalized sigma Weil model so obtained to the Poisson sigma model with target space  $M$  carrying a Hamiltonian Poisson action of the Poisson-Lie group  $G$ . This would yield a generalized Poisson-Weil sigma model and would be the gauging of the Poisson sigma model by the Poisson-Lie  $G$ -symmetry.<sup>4</sup>

The basic and equivariant Poisson-Lichnerowicz cohomology of  $M$  have been defined and studied by Ginzburg [42] also for this more general setting. Note that the moment map  $\mu$  would be  $G^\vee$ -valued rather than  $\mathfrak{g}^\vee$ -valued in this case, where  $G^\vee$  is dual Poisson-Lie dual group of  $G$ . The BV cohomology of the generalized Poisson-Weil sigma model should again be related to this more general cohomology.

The Poisson sigma model with a Poisson-Lie target space  $G$  has been studied in [51 – 53]. One may explore the relation of these models with the one resulting from the constructions just outlined.

It remains to be seen whether the generalized models are going to yield interesting topological field theories upon gauge fixing. All this is left to future work.

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<sup>4</sup>This possibility was suggested to us by F. Bonechi.



## A. The Weil model in components

In this appendix, we collect the explicit expressions of the action and the BV symmetry transformations of the Weil sigma model in terms of the component fields.

The expansion of the Weil sigma model superfields in components is given in (2.16). The ghost numbers of the components are given by the following table:

$$\begin{aligned}
 \text{gh } b_i &= 0, & \text{gh } A^+_i &= -1, & \text{gh } c^+_i &= -2, \\
 \text{gh } c^i &= +1, & \text{gh } A^i &= 0, & \text{gh } b^{+i} &= -1, \\
 \text{gh } B_i &= -1, & \text{gh } \psi^+_i &= -2, & \text{gh } C^+_i &= -3, \\
 \text{gh } C^i &= +2, & \text{gh } \psi^i &= +1, & \text{gh } B^{+i} &= 0.
 \end{aligned} \tag{A.1}$$

The Weil sigma model action  $S_W$  (cf. eq. (2.2)) in components reads

$$\begin{aligned}
 S_W = \int_{\Sigma} \Big[ & -b_i(F_A^i + B^{+i} - f^i_{jk}b^{+j}c^k) + A^+_i(D_A c^i - \psi^i) \\
 & + B_i(D_A \psi^i - f^i_{jk}c^j B^{+k} - f^i_{jk}b^{+j}C^k) - \psi^+_i(D_A C^i + f^i_{jk}c^j \psi^k) \\
 & + c^+_i \left( C^i - \frac{1}{2} f^i_{jk} c^j c^k \right) + C^+_i f^i_{jk} c^j C^k \Big],
 \end{aligned} \tag{A.2}$$

where

$$F_A^i = dA^i + \frac{1}{2} f^i_{jk} A^j A^k \tag{A.3}$$

is the curvature of the connection  $A$  and

$$D_A X^i = dX^i + f^i_{jk} A^j X^k, \tag{A.4a}$$

$$D_A Y_i = dY_i - f^k_{ji} A^j Y_k \tag{A.4b}$$

are the gauge covariant derivatives of  $X \in \Omega^*(\Sigma, \text{Ad } Q)$  and  $Y \in \Omega^*(\Sigma, \text{Ad}^\vee Q)$ , respectively. Above, the various fields are local forms on  $\Sigma$  obtained by the corresponding components of the basic superfields by the formal replacement  $\vartheta^\alpha \rightarrow dz^\alpha$ . Wedge multiplication of forms is understood.

The Weil sigma model BV variations (cf. eq. (2.4)) of the components are explicitly given by

$$\delta_W c^i = C^i - \frac{1}{2} f^i_{jk} c^j c^k, \tag{A.5a}$$

$$\delta_W A^i = \psi^i - D_A c^i, \tag{A.5b}$$

$$\delta_W b^{+i} = B^{+i} + F_A^i - f^i_{jk} c^j b^{+k}, \tag{A.5c}$$

$$\delta_W b_i = f^k_{ji} c^j b_k + f^k_{ji} C^j B_k, \tag{A.5d}$$

$$\delta_W A^+_i = D_A b_i + f^k_{ji} c^j A^+_k + f^k_{ji} C^j \psi^+_k - f^k_{ji} \psi^j B_k, \tag{A.5e}$$

$$\delta_W c^+_i = D_A A^+_i - f^k_{ji} b^{+j} b_k + f^k_{ji} c^j c^+_k \tag{A.5f}$$

$$- f^k_{ji} \psi^j \psi^+_k - f^k_{ji} B^{+j} B_k + f^k_{ji} C^j C^+_k,$$

$$\delta_W C^i = - f^i_{jk} c^j C^k, \tag{A.5g}$$

$$\delta_W \psi^i = -D_A C^i - f^i{}_{jk} c^j \psi^k, \quad (\text{A.5h})$$

$$\delta_W B^{+i} = D_A \psi^i - f^i{}_{jk} c^j B^{+k} - f^i{}_{jk} b^{+j} C^k, \quad (\text{A.5i})$$

$$\delta_W B_i = -b_i + f^k{}_{ji} c^j B_k \quad (\text{A.5j})$$

$$\delta_W \psi^+_i = -A^+_i + D_A B_i + f^k{}_{ji} c^j \psi^+_k, \quad (\text{A.5k})$$

$$\delta_W C^+_i = D_A \psi^+_i + f^k{}_{ji} c^j C^+_k - f^k{}_{ji} b^{+j} B_k - c^+_i, \quad (\text{A.5l})$$

as one can check by a simple computation.

## B. The Poisson-Weil model in components

In this appendix, we collect the explicit expressions of the action and the BV symmetry transformations of the Poisson-Weil sigma model in terms of the component fields.

The expansion of the Poisson-Weil sigma model superfields in terms of components is given in (2.16), (4.21). The ghost numbers of the components are given by table (A.1) and by the following one:

$$\begin{aligned} \text{gh } x^a &= 0, & \text{gh } \eta^{+a} &= -1, & \text{gh } y^{+a} &= -2, \\ \text{gh } y_a &= +1, & \text{gh } \eta_a &= 0, & \text{gh } x^+_a &= -1. \end{aligned} \quad (\text{B.1})$$

The Poisson-Weil sigma model action  $S_{\text{PW}}$  (cf. eq. (4.4)) in components is given by

$$\begin{aligned} S_{\text{PW}} = S_W + \int_{\Sigma} \left[ \eta_a D_A x^a + \frac{1}{2} P^{ab}(x) \eta_a \eta_b \right. \\ + \eta^{+a} (D_A y_a + \partial_a P^{bc}(x) \eta_b y_c + \partial_a u_i{}^b(x) \eta_b c^i - \partial_a \mu_i(x) \psi^i) \\ + \frac{1}{2} \eta^{+a} \eta^{+b} \left( \frac{1}{2} \partial_a \partial_b P^{cd}(x) y_c y_d + \partial_a \partial_b u_i{}^c(x) y_c c^i + \partial_a \partial_b \mu_i(x) C^i \right) \\ - u_i{}^a(x) y_a b^{+i} - \mu_i(x) B^{+i} + x^+_a (P^{ab}(x) y_b + u_i{}^a(x) c^i) \\ \left. - y^{+a} \left( \frac{1}{2} \partial_a P^{bc}(x) y_b y_c + \partial_a u_i{}^b(x) y_b c^i + \partial_a \mu_i(x) C^i \right) \right], \end{aligned} \quad (\text{B.2})$$

where  $S_W$  is given by (A.2) and

$$D_A x^a = dx^a - u_i{}^a(x) A^i, \quad (\text{B.3a})$$

$$D_A y_a = dy_a + \partial_a u_i{}^b(x) A^i y_b \quad (\text{B.3b})$$

are the gauge covariant derivatives of  $x$  and  $y$ , respectively. Recall that the various fields are local forms on  $\Sigma$  obtained from the corresponding components of the superfields by the formal replacement  $\vartheta^\alpha \rightarrow dz^\alpha$  and that wedge multiplication of forms is understood throughout. The main properties of the gauge covariant derivatives are collected in appendix C.

The Poisson-Weil sigma model BV variations (cf. eq. (4.7)) of the component fields are given by

$$\delta_{\text{PW}} c^i = \delta_W c^i, \quad (\text{B.4a})$$

$$\delta_{\text{PW}} A^i = \delta_W A^i, \quad (\text{B.4b})$$

$$\delta_{\text{PW}} b^{+i} = \delta_W b^{+i}, \quad (\text{B.4c})$$

$$\delta_{\text{PW}} b_i = \delta_W b_i - u_i^a(x) y_a, \quad (\text{B.4d})$$

$$\delta_{\text{PW}} A^+{}_i = \delta_W A^+{}_i - \partial_a u_i^b(x) \eta^{+a} y_b - u_i^a(x) \eta_a, \quad (\text{B.4e})$$

$$\begin{aligned} \delta_{\text{PW}} c^+{}_i &= \delta_W c^+{}_i - \partial_a u_i^b(x) \eta^{+a} \eta_b - u_i^a(x) x^+_a \\ &\quad - \frac{1}{2} \partial_a \partial_b u_i^c(x) \eta^{+a} \eta^{+b} y_c + \partial_a u_i^b(x) y^{+a} y_b, \end{aligned} \quad (\text{B.4f})$$

$$\delta_{\text{PW}} C^i = \delta_W C^i, \quad (\text{B.4g})$$

$$\delta_{\text{PW}} \psi^i = \delta_W \psi^i, \quad (\text{B.4h})$$

$$\delta_{\text{PW}} B^{+i} = \delta_W B^{+i}, \quad (\text{B.4i})$$

$$\delta_{\text{PW}} B_i = \delta_W B_i - \mu_i(x), \quad (\text{B.4j})$$

$$\delta_{\text{PW}} \psi^+{}_i = \delta_W \psi^+{}_i - \partial_a \mu_i(x) \eta^{+a}, \quad (\text{B.4k})$$

$$\delta_{\text{PW}} C^+{}_i = \delta_W C^+{}_i - \frac{1}{2} \partial_a \partial_b \mu_i(x) \eta^{+a} \eta^{+b} + \partial_a \mu_i(x) y^{+a}, \quad (\text{B.4l})$$

$$\delta_{\text{PW}} x^a = P^{ab}(x) y_b + u_i^a(x) c^i, \quad (\text{B.4m})$$

$$\delta_{\text{PW}} \eta^{+a} = D_A x^a + \partial_c P^{ab}(x) \eta^{+c} y_b + \partial_b u_i^a(x) \eta^{+b} c^i + P^{ab}(x) \eta_b, \quad (\text{B.4n})$$

$$\begin{aligned} \delta_{\text{PW}} y^{+a} &= -D_A \eta^{+a} - \partial_c P^{ab}(x) \eta^{+c} \eta_b \\ &\quad - \frac{1}{2} \partial_c \partial_d P^{ab}(x) \eta^{+c} \eta^{+d} y_b - \frac{1}{2} \partial_b \partial_c u_i^a(x) \eta^{+b} \eta^{+c} c^i \\ &\quad - P^{ab}(x) x^+_b + \partial_c P^{ab}(x) y^{+c} y_b + \partial_b u_i^a(x) y^{+b} c^i + u_i^a(x) b^{+i}, \end{aligned} \quad (\text{B.4o})$$

$$\delta_{\text{PW}} y_a = \frac{1}{2} \partial_a P^{bc}(x) y_b y_c + \partial_a u_i^b(x) y_b c^i + \partial_a \mu_i(x) C^i, \quad (\text{B.4p})$$

$$\begin{aligned} \delta_{\text{PW}} \eta_a &= D_A y_a + \frac{1}{2} \partial_a \partial_d P^{bc}(x) \eta^{+d} y_b y_c + \partial_a P^{bc}(x) \eta_b y_c \\ &\quad + \partial_a \partial_c u_i^b(x) \eta^{+c} y_b c^i + \partial_a u_i^b(x) \eta_b c^i \\ &\quad - \partial_a \mu_i(x) \psi^i + \partial_a \partial_b \mu_i(x) \eta^{+b} C^i, \end{aligned} \quad (\text{B.4q})$$

$$\begin{aligned} \delta_{\text{PW}} x^+{}_a &= D_A \eta_a + \frac{1}{2} \partial_a P^{bc}(x) \eta_b \eta_c - \partial_a \partial_c u_i^b(x) \eta^{+c} y_b A^i \\ &\quad + \partial_a \partial_c u_i^b(x) \eta^{+c} \eta_b c^i + \partial_a \partial_d P^{bc}(x) \eta^{+d} \eta_b y_c \\ &\quad + \frac{1}{4} \partial_a \partial_d \partial_e P^{bc}(x) \eta^{+d} \eta^{+e} y_b y_c + \frac{1}{2} \partial_a \partial_c \partial_d u_i^b(x) \eta^{+c} \eta^{+d} y_b c^i \\ &\quad + \partial_a P^{bc}(x) x^+_b y_c + \partial_a u_i^b(x) x^+_b c^i - \frac{1}{2} \partial_a \partial_d P^{bc}(x) y^{+d} y_b y_c \\ &\quad - \partial_a \partial_c u_i^b(x) y^{+c} y_b c^i - \partial_a u_i^b(x) y_b b^{+i} - \partial_a \partial_b \mu_i(x) \eta^{+b} \psi^i \\ &\quad + \frac{1}{2} \partial_a \partial_b \partial_c \mu_i(x) \eta^{+b} \eta^{+c} C^i - \partial_a \partial_b \mu_i(x) y^{+b} C^i - \partial_a \mu_i(x) B^{+i}, \end{aligned} \quad (\text{B.4r})$$

where the Weil sigma model  $\delta_W$  variations are given by (A.5) and

$$D_A \eta^{+a} = d\eta^{+a} - \partial_b u_i^a(x) A^i \eta^{+b}, \quad (\text{B.5a})$$

$$D_A \eta_a = d\eta_a + \partial_a u_i^b(x) A^i \eta_b \quad (\text{B.5b})$$

are the gauge covariant derivatives of  $\eta^+$  and  $\eta$ , respectively.<sup>5</sup>

<sup>5</sup>Here, we are abusing our terminology. Strictly speaking,  $D_A \eta$ , as defined above, is gauge covariant only when  $y$  vanishes. See again appendix C.

### C. $G$ and general covariance

The covariance of the fields of the sigma models studied in the main body of the paper is rather intricate. The embedding field is not simply a map  $x : \Sigma \rightarrow M$ , as in the ordinary ungauged sigma models, but a section of the bundle  $E_M = Q \times_G M$ , whose definition combines in a non trivial manner the principal  $G$ -bundle  $Q$  on  $\Sigma$  and the manifold  $M$  with  $G$ -action. The other fields are sections of bundles which are (related to) pull-backs by  $x$  of bundles on  $E_M$ . The construction of suitable gauge and general covariant derivatives of the fields is thus a subtle matter. For the sake of concreteness, it may be useful to write down the covariance of these fields and their covariant derivatives in terms of the cocycles representing the bundles, which they are sections of. This is done in the present appendix. More material on this topic can be found for instance in [18–20, 50].

Let  $G$  be a connected Lie group. Let the manifold  $M$  carry a left  $G$ -action. The fundamental vector field  $u$  of the  $G$ -action is defined by the relation

$$g^a(m) = m^a - \xi^i u_i^a(m) + O(\xi^2), \quad (\text{C.1})$$

for  $g = \exp(\xi) \in G$  with  $\xi \in \mathfrak{g}$ .  $u$  is  $G$ -equivariant, i.e. for  $g \in G$ ,

$$\partial_b g^{-1a} \circ g u_i^b \circ g = (\text{Ad}g)^j_i u_j^a. \quad (\text{C.2})$$

Let  $Q$  be a principal  $G$ -bundle on the closed surface  $\Sigma$ . Let  $\{g_{AB}(z)\}$  be a  $G$ -valued 1-cocycle representing  $Q$ . Here,  $A, B, C \dots$  are local trivialization indices. The 1-cocycle condition

$$g_{AB}(z)g_{BC}(z) = g_{AC}(z), \quad (\text{C.3})$$

when defined, holds.

Let  $E_M$  be the fiber bundle on  $\Sigma$  represented by the non linear cocycle  $\{g_{AB}^a(z, m_B)\}$  obtained from the  $G$ -valued 1-cocycle  $\{g_{AB}(z)\}$  representing  $Q$  via the  $G$ -action on  $M$ . A section  $x \in \Gamma(\Sigma, E_M)$  is given locally as a collection of maps  $\{x_A(z)\}$  into  $M$  matching as

$$x_A^a(z) = g_{AB}^a(z, x_B(z)). \quad (\text{C.4})$$

Let  $x \in \Gamma(\Sigma, E_M)$ . Let  $x^* \text{Vert} TE_M$  be the vector bundle on  $\Sigma$  represented by the 1-cocycle  $\{C_{AB}^a{}_b(z)\}$ , where  $C_{AB}^a{}_b(z) = \partial_b g_{AB}^a(z, x_B(z))$ . A section  $v \in \Omega^0(\Sigma, x^* \text{Vert} TE_M)$  is given locally as a collection of  $TM$ -valued functions  $\{v_A^a(z)\}$  matching as

$$v_A^a(z) = \partial_b g_{AB}^a(z, x_B(z)) v_B^b(z). \quad (\text{C.5})$$

In similar fashion, let  $x^* \text{Vert}^* TE_M$  be the vector bundle on  $\Sigma$  represented by the 1-cocycle  $\{C^*{}_{ABa}{}^b(z)\}$ , where  $C^*{}_{ABa}{}^b(z) = \partial_a g_{BA}^b(z, x_A(z))$ . A section  $s \in \Omega^0(\Sigma, x^* \text{Vert}^* TE_M)$  is given locally as a collection of  $T^*M$ -valued functions  $\{s_{Aa}(z)\}$  matching as

$$s_{Aa}(z) = \partial_a g_{BA}^b(z, x_A(z)) s_{Bb}(z). \quad (\text{C.6})$$

We want to construct gauge covariant derivatives for sections of the bundles  $E_M$ ,  $x^* \text{Vert} TE_M$ ,  $x^* \text{Vert}^* TE_M$ . To this end, one needs a connection of  $Q$ . Recall that a connection  $A$  of  $Q$  is given locally as a collection of  $\mathfrak{g}$ -valued 1-forms  $\{A_A^i(z)\}$  matching as

$$A_A^i(z) = (\text{Ad}g_{AB}(z))^i_j A_B^j(z) + (g_{AB}(z) dg_{AB}(z)^{-1})^i, \quad (\text{C.7})$$

where here and below  $d$  denote the de Rham differential of  $\Sigma$ .

The following relation

$$dg_{AB}{}^a(z, m_B) = (g_{AB}(z)dg_{AB}(z)^{-1})^i u_i{}^a(g_{AB}(z, m_B)) \quad (\text{C.8})$$

plays a basic role in the following analysis of covariance.

For  $x \in \Gamma(\Sigma, E_M)$ , define

$$D_A x^a = dx^a - u_i{}^a(x)A^i. \quad (\text{C.9})$$

Using (C.7), (C.8), one finds that

$$(D_A x)_A{}^a(z) = \partial_b g_{AB}{}^a(z, x_B(z))(D_A x)_B{}^b(z). \quad (\text{C.10})$$

This shows that  $D_A x \in \Omega^1(\Sigma, x^* \text{Vert } TE_M)$ . In this sense,  $D_A x$  is the gauge covariant derivative of  $x$ .

Let  $x \in \Gamma(\Sigma, E_M)$ . For  $v \in \Omega^0(\Sigma, x^* \text{Vert } TE_M)$ , define

$$D_A v^a = dv^a - \partial_b u_i{}^a(x)A^i v^b. \quad (\text{C.11})$$

Then, using (C.7), (C.8), one finds

$$\begin{aligned} (D_A v)_A{}^a(z) &= \partial_b g_{AB}{}^a(z, x_B(z))(D_A v)_B{}^b(z) \\ &\quad + \partial_b \partial_c g_{AB}{}^a(z, x_B(z))(D_A x)_B{}^b(z)v_B{}^c(z). \end{aligned} \quad (\text{C.12})$$

Similarly, for  $s \in \Omega^0(\Sigma, x^* \text{Vert }^* TE_M)$ , define

$$D_A s_a = ds_a + \partial_a u_i{}^b(x)A^i s_b. \quad (\text{C.13})$$

Then, using (C.7), (C.8) again, one obtains

$$\begin{aligned} (D_A s)_{Aa}(z) &= \partial_a g_{BA}{}^b(z, x_A(z))(D_A s)_{Bb}(z) \\ &\quad + \partial_a \partial_b g_{BA}{}^c(z, x_A(z))(D_A x)_A{}^b(z)s_{Bc}(z). \end{aligned} \quad (\text{C.14})$$

Note that  $D_A v \notin \Omega^1(\Sigma, x^* \text{Vert } TE_M)$  because of the second term in the right hand side of (C.12). However, notice that this term would be absent if  $M$  were a linear space and the  $G$ -action on  $M$  were linear, that is if  $E_M$  were a vector bundle. For this reason, with an abuse of language, we call  $D_A v$  the gauge covariant derivative of  $v$ . Similarly,  $D_A s \notin \Omega^1(\Sigma, x^* \text{Vert }^* TE_M)$  because of the second term in the right hand side of (C.14). Again, with an abuse of language, we call  $D_A s$  the gauge covariant derivative of  $s$ .

One can correct the lack of full covariance found above by using a  $G$ -invariant connection of  $M$ . Recall that a connection  $\Gamma$  of  $TM$  is said  $G$ -invariant, if, for any  $g \in G$

$$\Gamma^a{}_{bc} = \Gamma^d{}_{ef} \circ g \partial_d g^{-1a} \circ g \partial_b g^e \partial_c g^f + \partial_d g^{-1a} \circ g \partial_b \partial_c g^d. \quad (\text{C.15})$$

The Levi-Civita connection associated to a  $G$ -invariant Riemannian metric is  $G$ -invariant.

For a section  $v \in \Omega^0(\Sigma, x^* \text{Vert } TEM)$ , we define

$$D_{\nabla A} v^a = D_A v^a + \Gamma^a_{bc}(x) D_A x^b v^c. \quad (\text{C.16})$$

Then, under a change of local trivialization

$$(D_{\nabla A} v)_A^a(z) = \partial_b g_{AB}^a(z, x_B(z)) (D_{\nabla A} v)_B^b(z). \quad (\text{C.17})$$

Thus,  $D_{\nabla A} v \in \Omega^1(\Sigma, x^* \text{Vert } TEM)$  and  $D_{\nabla A} v$  is a genuine covariant derivative. Similarly, for a section  $s \in \Omega^0(\Sigma, x^* \text{Vert }^* TEM)$ , we define

$$D_{\nabla A} s_a = D_A s_a - \Gamma^c_{ba}(x) D_A x^b s_c. \quad (\text{C.18})$$

Then, under a change of local trivialization

$$(D_{\nabla A} s)_{Aa}(z) = \partial_a g_{BA}^b(z, x_A(z)) (D_{\nabla A} s)_{Bb}(z). \quad (\text{C.19})$$

Thus,  $D_{\nabla A} s \in \Omega^1(\Sigma, x^* \text{Vert }^* TEM)$  and  $D_{\nabla A} s$  is a genuine covariant derivative.

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